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MAGNETO CONVECTIVE INSTABILITIES DRIVEN BY INTERNAL UNIFORM VOLUMETRIC HEATING

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A linear stability analysis is performed to investigate the onset of convective motions in a flat cavity filled with liquid metal in which a volumetric heat source is distributed uniformly and a horizontal magnetic field is imposed. A quasi-2D mathematical model is derived by integrating the 3D governing equations along the magnetic field direction, which yields a dissipation term in the 2D equations that accounts for 3D viscous effects in thin boundary layers at walls perpendicular to the field. This type of buoyant flow without magnetic field has been investigated by Roberts [1] and the present study extends those results to magnetohydrodynamic conditions. Numerical simulations are performed to support the analytical results and to describe the main convective flow patterns.

Introduction. Thermal convective motion produced by uniform internal heat sources in a liquid metal layer, as analyzed in the present study, is a fundamental heat transfer problem of interest for engineering applications such as, e.g., nuclear fusion reactors. Here a plasma is confined in a torus by means of a strong magnetic field. Neutron heat is removed by a liquid metal circulating in the so-called blanket. Most of the nuclear power is deposited in the liquid metal leading to significantly non-uniform thermal conditions that result in complex convective flow patterns that are affected by the magnetic field [2, 3].

The problem of natural convection driven by a temperature difference across a fluid layer, the so-called Rayleigh-Bénard convection, has been extensively analyzed for applications in crystal growth technology. When the fluid is heated from below, it remains motionless until the temperature difference, quantified by the non-dimensional Rayleigh number Ra, exceeds a critical value $\mathrm{Ra}_{\mathrm{cr}}$ and then thermal convection sets in. Chandrasekhar [4] shows that by applying a magnetic field instabilities occur at higher values of Ra compared to hydrodynamic conditions. This means that the induced electromagnetic forces tend to stabilize the flow. At marginal stability convection appears in the form of rolls aligned with the horizontal component of the magnetic field. Analytical and experimental investigations of MHD Bénard-convection can be found in [5]. When a strong magnetic field is applied, electromagnetic Lorentz forces elongate vortices along magnetic field lines and force the fluid to move in planes perpendicular to the field, while the motion along field lines is damped [6]. This leads to a quasi-two dimensional (Q2D) MHD flow, where dissipation losses, due to Joule and viscous effects, are localized in thin Hartmann boundary layers along the walls perpendicular to the magnetic field. An explanation of dissipative effects in Hartmann layers is given in [7, 8]. Q2D models reduce the basic governing equations to a 2D problem by analytical integration along magnetic field lines. In the 2D equations, 3D MHD effects are modelled by a term that accounts for viscous and Joule dissipation in Hartmann layers. Those approaches are used to investigate problems related to fusion blankets, where intense magnetic fields are present [9].

In the problem studied in this paper, convective motions are driven by non-uniform thermal conditions caused by heat sources distributed in a fluid. The steady laminar hydrodynamic convection in an infinite horizontal fluid layer confined between an isothermal upper plate and a lower one that is thermally insulating has been studied by different authors. This configuration differs from Bénard-convection since temperature boundary conditions are asymmetric and the vertical temperature profile in the motionless state is parabolic rather than linear. Experimental studies of instabilities in a horizontal fluid layer heated uniformly are described in [10]. Roberts [1] carried out a stability analysis that showed that convective motions occurr at $\mathrm{Ra}_{\mathrm{cr}} \simeq 2772$ in the form of marginally stable rolls. Thirlby [11] performed a numerical analysis and determined the parameters at which polygonal cells and rolls occur in hydrodynamic flows.

The aim of the present study is investigating the influence of a horizontal magnetic field on the onset of instabilities in liquid metal flows with volumetric thermal sources and identifying the main convective patterns. The geometrical configuration chosen for this study is the one used in [1]. Model equations describing the Q2D MHD convective flow are derived (Section 2) and a linear stability analysis is performed (Section 3) to determine the onset of convection depending on the intensity of the applied heat source and the strength of the magnetic field. A better understanding of the features of convective flow patterns is obtained by means of numerical simulations.

1. Formulation of the problem and governing equations. Let us consider an electrically conducting fluid, such as a liquid metal, filling a horizontal shallow cavity, in which a volumetric heat source q is uniformly distributed. The top wall at $y = H$ is isothermal, the bottom at $y = 0$ adiabatic, $\partial T/\partial y = 0$, and the Hartmann walls at $z = \pm A$, perpendicular to the magnetic field, are adiabatic, $\partial T/\partial z = 0$ (Fig. 1).

Density changes due to temperature variation are restricted to the buoyancy term $\rho\beta\,(T - T_{\mathrm{ref}})\,\mathbf{g}$, according to the Boussinesq approximation. Here ρ is the density at the reference temperature T_{ref}, β the volumetric thermal expansion coefficient and $\mathbf{g} = -g\hat{\mathbf{y}}$ is the gravitational acceleration. The non-dimensional equations governing the problem account for the balance of momentum, conservation of mass and charge, and the current density is determined by the Ohm's law:

$$\frac{1}{\mathrm{Pr}}\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right) = -\nabla p + \nabla^2\mathbf{v} + \mathrm{Ra}T\hat{\mathbf{y}} + \mathrm{Ha}^2\,(\mathbf{j}\times\mathbf{B})\,, \tag{1}$$

$$\nabla\cdot\mathbf{v} = 0, \tag{2}$$

$$\nabla\cdot\mathbf{j} = 0, \tag{3}$$

$$\mathbf{j} = -\nabla\phi + \mathbf{v}\times\mathbf{B}. \tag{4}$$

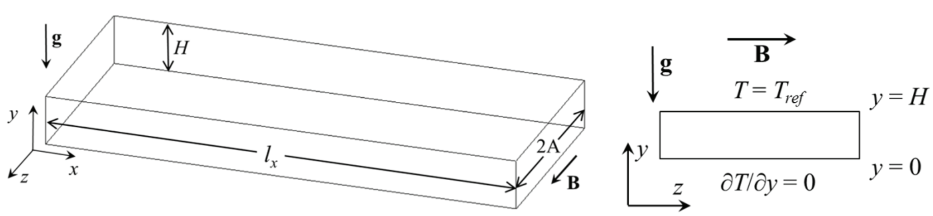

Fig. 1. Sketch of the geometry and reference system. Walls perpendicular to the magnetic field are thermally insulating, the top wall of the cavity is kept at a constant temperature; the bottom is adiabatic. Periodic conditions are assumed in x-direction. A constant magnetic field is applied in horizontal z-direction.

The temperature distribution is given by the energy balance equation

$$\frac{\partial T}{\partial t} + (\mathbf{v} \cdot \nabla)T = \nabla^2 T + 1. \tag{5}$$

The dimensional volumetric heat source q has been scaled by $\lambda \Delta T / H^2$ and normalized to unity by defining the characteristic temperature difference as $\Delta T = qH^2/\lambda$. The dimensionless variables $\mathbf{v}$, t, $\mathbf{j}$, ϕ and $\mathbf{B}$ are obtained by scaling velocity, time, electric current density, electric potential and magnetic field by the reference quantities $v_0 = \alpha/H$, H^2/α, $\sigma v_0 B_0$, $v_0 B_0 H$ and B_0, respectively. The typical length scale H is the distance between horizontal walls. Thermal diffusivity $\alpha = \lambda/(\rho c_p)$, thermal conductivity λ, specific heat c_p, kinematic viscosity ν and electrical conductivity σ are assumed to be constant in the temperature range considered. The non-dimensional temperature T is given by $(T^* - T_{\mathrm{ref}})/\Delta T$, where T^* is the local dimensional temperature. The dimensionless parameters that control the flow are the Prandtl number Pr, the Rayleigh number Ra and the Hartmann number Ha:

$$\mathrm{Pr} = \frac{\nu}{\alpha}, \quad \mathrm{Ra} = \frac{g\beta q H^5 \rho c_p}{\nu \lambda^2}, \quad \mathrm{Ha} = B_0 H \sqrt{\frac{\sigma}{\rho\nu}}. \tag{6}$$

The Prandtl number represents the rate of momentum diffusion to the one of heat diffusion. The Rayleigh number describes the intensity of the applied heating. The Hartmann number gives a non-dimensional measure for the strength of the magnetic field. In order to quantify the magnitude of convection, we introduce a quantity defined as the ratio between mean temperature differences across the fluid layer without motion and with convection [1, 11]:

$$M = \frac{\int_V T_{\mathrm{cond}} \mathrm{d}V}{\int_V T_{\mathrm{conv}} \mathrm{d}V} = \frac{1}{3\overline{\overline{T}}}. \tag{7}$$

2. 2D model equations. The procedure followed to obtain the equations for the Q2D model is analogous to the one used in [5, 7, 8]. Starting from the 3D equations (Section 1), an equation for vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{v}$ is derived. For the given boundary conditions and an applied magnetic field in z-direction the flow is characterized by a Q2D velocity $\mathbf{v} = (u, v, 0)$ and vorticity $\boldsymbol{\omega} = (0, 0, \omega)$, where u, v, ω may depend on (x, y, z):

$$\frac{1}{\mathrm{Pr}} \left(\partial_t \omega + u \partial_x \omega + v \partial_y \omega\right) = \nabla^2 \omega + \mathrm{Ha}^2 \partial_z j_z + \mathrm{Ra}\, \partial_x T. \tag{8}$$

In a quasi 2D flow, the velocity and vorticity can be expressed by a separation ansatz, e.g., $u = \hat{u}(t, x, y) f(z)$ By integrating the vorticity equation along magnetic field lines, with no slip at Hartmann walls, $f(z = \pm a = \pm A/H) = 0$, and thin wall condition [12], we obtain

$$\frac{1}{\mathrm{Pr}} \left(\partial_t \hat{\omega} + \hat{u} \partial_x \hat{\omega} + \hat{v} \partial_y \hat{\omega}\right) = \nabla^2_{xy} \hat{\omega} - \frac{1}{\tau} \hat{\omega} + \mathrm{Ra}\, \partial_x T \quad \text{with} \quad \frac{1}{\tau} = \frac{\mathrm{Ha}}{a} + \frac{c\mathrm{Ha}^2}{a + c}, \tag{9}$$

where $c = \sigma_{\mathrm{w}} t_{\mathrm{w}} / (\sigma H)$ is the conductance parameter that gives the ratio of the conductance of the wall material with thickness t_{w} and electrical conductivity σ_{w} to the one of the fluid. In Eq. (9), terms on the left-hand side represent the convective transport of vorticity and its time variation, on the right-hand side there are two dissipation terms. The first one describes viscous losses due to gradients of vorticity in a plane perpendicular to $\mathbf{B}$. The term $-\hat{\omega}/\tau$ represents viscous dissipation in the Hartmann layers and Joule dissipation in layers and in the thin electrically conducting wall. The dissipation factor $1/\tau$ is related to a typical decay time of the vorticity [7]. For electrically insulating Hartmann walls ($c = 0$), we find $1/\tau \to \mathrm{Ha}/a$, and for perfectly conducting walls ($c = \infty$), $1/\tau \to \mathrm{Ha}^2$, namely, in ducts with highly electrically conducting walls a rapid damping of perturbations occurs.

3. Linear stability analysis. In the problem studied, the basic steady state is motionless and there is a parabolic temperature distribution along the vertical coordinate y (see Fig. 1). When the internal heat source, i.e. the Rayleigh number Ra, is large enough, the base state loses its stability due to increased buoyancy forces that are not balanced anymore by viscous effects. Convective motions occur, whose intensity depends on the strength of the internal heat source (Ra) and on the magnitude of the applied magnetic field (Ha). A linear stability analysis is performed that consists in following the evolution of small perturbations applied to the equilibrium state by linearizing equations (5) and (9). The stability is determined by solving the resulting eigenvalue problem. In order to derive disturbance equations, temperature, velocity and vorticity are decomposed as the sum of a basic state denoted by the subscript '0' and a perturbation indicated by prime, e.g., $T = T_0 + \varepsilon T'$, where ε is a small parameter. Those expressions are introduced into Eqs. (5) and (9), and the terms of $O(\varepsilon^2)$ are neglected in the small perturbation limit. We expand perturbations in normal modes as, e.g., $T' = i\Theta(y)\mathrm{e}^{st+ikx}$, where k is a real horizontal wavenumber, s is the temporal rate of growth of the perturbation. In order to satisfy mass conservation (2), we introduce a stream function $\psi'(x,y)$, such that $\mathbf{v}' = \nabla \times (\psi' \cdot \hat{\mathbf{z}})$ and $\omega' = -\nabla^2\psi'$. After some mathematical work, the equations describing the stability of the problem become:

$$\begin{aligned} \left(D^2 - k^2 - \frac{1}{\tau} - \frac{s}{\mathrm{Pr}}\right)\Omega - k\mathrm{Ra}\,\Theta &= 0, \\ \left(D^2 - k^2 - s\right)\Theta - ky\Psi &= 0 \\ \left(D^2 - k^2\right)\Psi + \Omega &= 0, \end{aligned} \tag{10}$$

where $D^2 = \partial^2/\partial y^2$ and Ω, Θ, Ψ are the amplitude functions of vorticity, temperature and stream function perturbations, respectively. A numerical procedure has been implemented in Matlab, where finite difference techniques are used for the solution of the eigenvalue problem.

For given (k, Ra), a generalized discrete eigenvalue problem $\mathbf{A}(k, \mathrm{Ra})\mathbf{x} = s\mathbf{B}\mathbf{x}$ is solved. The eigenvector $\mathbf{x}$ is composed by Ω, Θ and Ψ at corresponding grid points. The solution procedure is as follows. For a given wave number k, the Rayleigh number Ra is varied till a value is reached for which an eigenvalue s exists with a real part equal to zero, i.e. until the solution reaches the stability limit. This couple (k, Ra) represents a point on the neutral curve $\mathrm{Ra}(k)$ (see Fig. 2). Instability sets in for $\mathrm{Ra}_{\mathrm{cr}} = \min(\mathrm{Ra}(k))$ at the corresponding critical wave number k_{cr}.

4. Results. For $1/\tau \to 0$, the problem is equivalent to the hydrodynamic flow ($\mathrm{Ha} = 0$) considered in [1], for which $\mathrm{Ra}_{\mathrm{cr}} = 2772$ and $k_{\mathrm{cr}} = 2.63$ are predicted. This case is first investigated to validate the used numerical model. In a second step, a uniform horizontal magnetic field is imposed and its influence on the stability of the considered magneto-convective flow is studied. Numerical simulations are also performed both to confirm the linear stability analysis and to complement the results by means of 3D and Q2D nonlinear solutions for $\mathrm{Ra} > \mathrm{Ra}_{\mathrm{cr}}$.

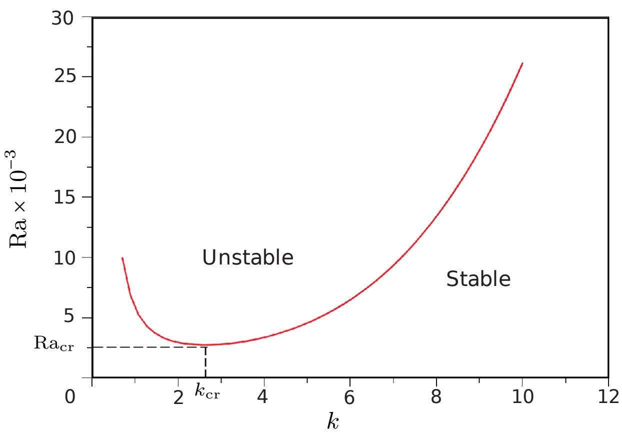

Fig. 2. Neutral stability curve for hydrodynamic flow (Ha = 0) showing the marginal Rayleigh number as a function of the wavenumber.

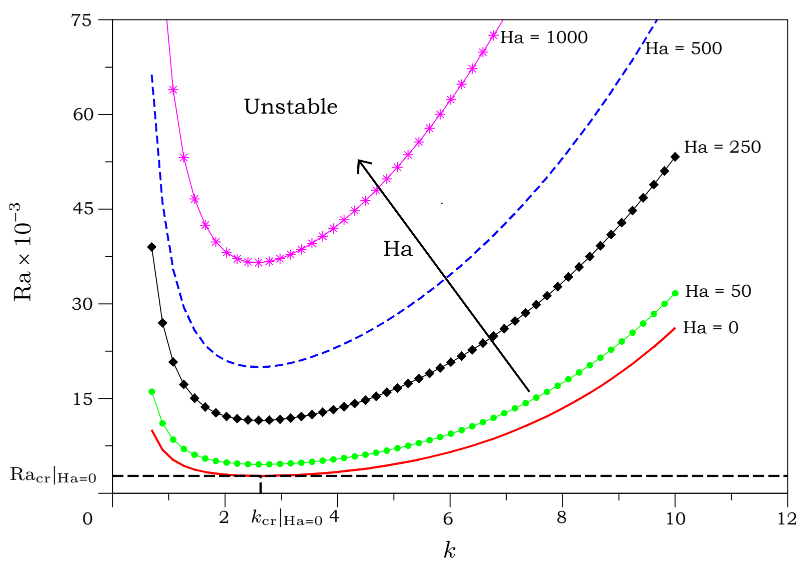

Fig. 3. Neutral stability curves for the case of electrically insulating walls $c = 0$ and various Hartmann numbers Ha.

Let us consider now magneto-convective flows in an electrically insulating cavity ($c = 0$) with $A/H = 2$. We analyze the influence of the magnetic field strength (Ha) on the flow stability. In Fig. 3, neutral stability curves are depicted for various Ha. The curve for the hydrodynamic case (Ha = 0) is shown for comparison. It can be seen that, as expected [4], the magnetic field stabilizes the flow, i.e. by increasing Ha the onset of convection occurs at higher values of Ra.

In the following, we fix the Hartmann number to Ha = 200 ($1/\tau = 100$) and we determine by means of numerical simulations nonlinear solutions when $\mathrm{Ra} > \mathrm{Ra_{cr}}$. Calculations are performed by using the Q2D model described in Section 2 that has been implemented in the finite volume code OpenFOAM. The axial length l_x (see Fig. 1) was chosen such that 8 convective cells fit in the computational domain at $\mathrm{Ra_{cr}}$.

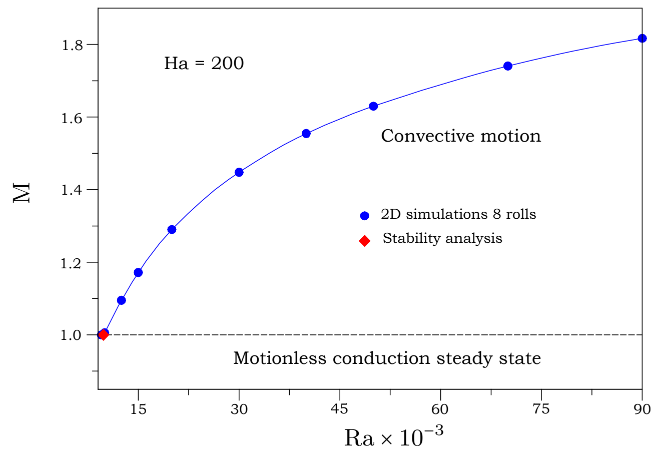

Fig. 4. Modified Nusselt number M (7) that quantifies the strength of convective motion as a function of the Rayleigh number.

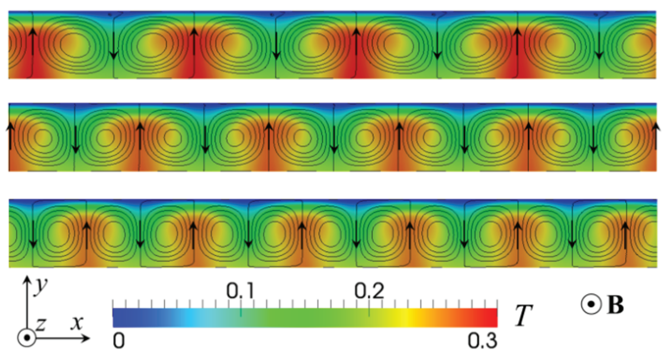

Fig. 5. Contours of scaled temperature and electric potential isolines (non-dimensional distance 0.9) for Q2D MHD flow at Ha = 200 and Ra = 70000.

In order to quantify the intensity of the convective motion, the quantity M (7) is calculated and plotted as a function of Ra in Fig. 4. The numerically predicted critical Rayleigh number $\mathrm{Ra_{cr}} = \mathrm{Ra(M = 1)}$ agrees very well with the one obtained by the stability analysis, $\mathrm{Ra_{cr}} = 9821.3$. By increasing the Rayleigh number, M becomes larger, i.e. the convective heat transfer intensifies. For sufficiently large Ra, various solutions coexist characterized by 8, 10 and 12 rolls. An example is shown in Fig. 5 for the flow at Ra = 70000. Here contours of the temperature T and isolines of the electric potential, which serve as approximate streamlines, are depicted on the middle cross-section of the cavity.

In Fig. 6, contours of the temperature and isolines of the electric potential are compared for three Rayleigh numbers. When the heating becomes stronger, the motion intensifies, as indicated by a larger number of potential isolines. When increasing the Rayleigh number, the heat transfer at the upper wall is enhanced by stronger convective motions and this leads to smaller non-dimensional temperature in the fluid. In Fig. 7, the vertical component of the velocity is plotted for various Ra along the axial coordinate on a line at $y = 0.5$. When approaching the stability limit ($\mathrm{Ra_{cr}} = 9821.3$), the velocity profile resembles a harmonic function. By rising Ra, additional modes appear due to nonlinear interactions leading to more complex velocity profiles.

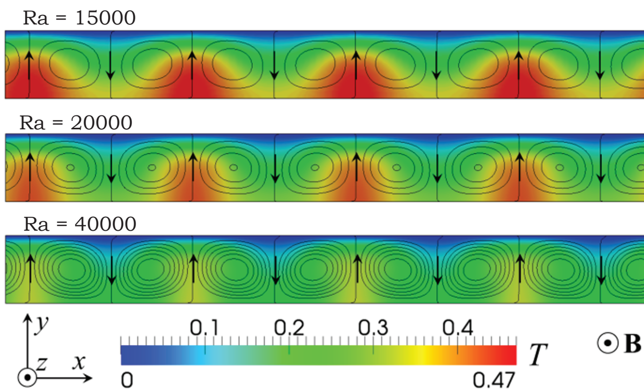

Fig. 6. Contours of scaled temperature and electric potential isolines (non-dimensional distance 0.56) for Q2D MHD flows at $Ha = 200$ and different Ra.

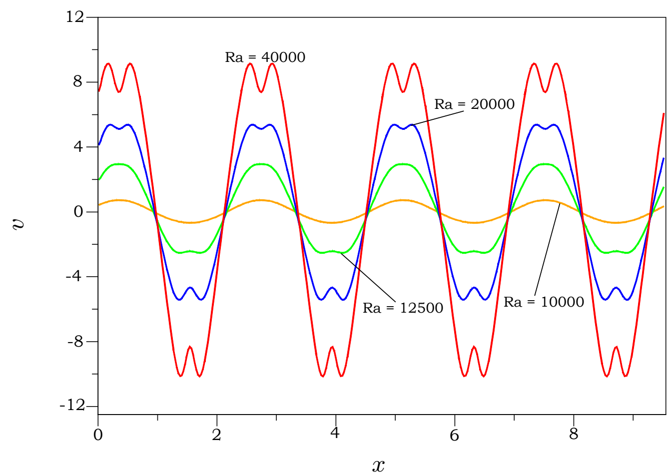

Fig. 7. Vertical velocity as a function of the coordinate x for various Ra ($Ha = 200$).

5. Conclusions. Magneto-convection caused by a uniform volumetric heat source distributed in a liquid metal layer has been investigated to identify the influence of an imposed magnetic field on the flow stability and on convective patterns. A linear stability analysis is performed based on a Q2D model and the critical Rayleigh number for the onset of convection has been calculated for increasing Hartmann numbers Ha, i.e. for progressively stronger magnetic fields. The occurrence of convection is delayed when Ha becomes larger. For supercritical conditions, solutions with different wavenumbers coexist. At the onset of convection, velocity perturbations occur first as harmonic functions along x. By increasing Ra, higher modes appear due to nonlinear interactions. Three dimensional simulations indicate the validity of the Q2D model for moderate internal volumetric heating. When Ra is sufficiently large, deviations from a 2D behavior are visible along the magnetic field lines.

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