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DYNAMO EQUATIONS WITH RANDOM COEFFICIENTS

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Galactic dynamo is caused by two effects. One of them is caused by differential rotation and the other is determined by turbulent motions. In some galaxies there is a strong star formation or other processes which are connected with local regions of hot gas. Turbulent motions in such zones differ from the those in warm gas. It is useful to model such processes with dynamo equations that contain random coefficients. The coefficient of alpha-effect can take two different values. The first one is related to warm gas and it is the same as the coefficient for most of the galaxies studied before. The second one characterizes hot gas, which can be connected with the star formation or other processes with high velocities and large portions of energy. This coefficient is random and changes with a time, which is much less than typical times of galactic dynamo. The probability to obtain the second value of the coefficient is determined by the intensity of the star formation. We have obtained some critical values of probability, for which dynamo cannot support the magnetic field growth. Also we have calculated average velocities of the magnetic field growth and its dispersion. For calculations, we used both numerical and asymptotical methods.

Introduction. It is believed that the generation of magnetic fields of galaxies and other astrophysical objects is described by the so-called dynamo theory [1]. The dynamo mechanism is usually based on two effects: differential rotation and the alpha-effect. Differential rotation is determined by non-solid body rotation of galaxies: the linear velocity of their rotation is constant at large distances from the galaxy center [2]. The alpha-effect describes turbulent motions of the interstellar medium. Each of them describes only the magnetic field decay, but their joint action can cause its exponential growth [3]. The magnetic field generation is a threshold process: if the so-called dynamo number (that includes the typical velocity of the interstellar gas, half-thickness of the galaxy disc and the angular velocity of the galaxy rotation) is larger than some value, the field can be generated, else it decays [4].

The alpha-effect and the differential rotation can be described by dimensionless coefficients R_α and R_ω. These coefficients usually include some averaged characteristics of the interstellar medium, such as turbulent velocities, viscosity, etc. [4]. For "calm" galaxies, where there are no intensive processes and the concentration of ionized hydrogen is of an order of 10^{-1} [5], these parameters describe the magnetic field growth well. But if there are some intensive processes in the galaxy, such as star formation or supernovae explosions (which create regions of ionized hydrogen), the turbulent motions including the alpha-effect will be changed.

A possible solution of this problem is a model with a random coefficient of the alpha-effect [6]. Basic aspects of dynamo with fluctuating α have been described by Proctor [7]. Moreover, such model for geomagnetic dynamo was also considered by Stefani and Gerbeth [8]. As for galactic dynamos with random coefficients, the model with local approach was described by Sur and Subramanian [9]. Although, some details are still not clear. For example, it is important to take into account that the alpha-effect can differ for various parts of the galaxy. Then, it would be important to make some asymptotic estimates of the magnetic field growth that are not connected with the numerical solution with non-ideal computer generators of random numbers.

We consider the alpha-effect parameter as a random process that takes two different values. The first value is connected with warmed atomic hydrogen, and the second one describes turbulent motions in regions with a highly ionized hot gas. The second option occurs with a probability p, which is determined by the ratio between hot and warm gas components.

We study the magnetic field of galaxies using the so-called no-z model. It is adopted such that the magnetic field disc is quite thin, so we can change z-derivatives of the magnetic field components by algebraic expressions [10]. One of the coefficients in this model is random. We use different types of equations. First, we use a linear model that contains a system of two ordinary equations. For this model, we obtain some numerical results. The main feature of them is the intermittency: higher momenta of the solution grow faster than lower ones [11]. Then, we obtain some asymptotic estimates of the magnetic field growth, using the invariant measure technique [12]. The asymptotic results are quite close to the numerical ones.

The magnetic field generation is connected with the transition of the kinetic energy of turbulent motions to the magnetic field energy, and the magnetic field growth is limited by some equipartition value. So, we also use a nonlinear type of the system of dynamo equations. The magnetic field growth for small values is quite similar to the linear case, but after the growth saturation the magnetic field does not stabilize. It has some oscillations near the equipartition value.

Then, we should take into account the dissipation of the magnetic field, so we also calculate the magnetic field using a model with partial differential equations.

1. Governing equations. To describe the magnetic field of the galaxy, we use equations of the no-z model [13]:

$$\begin{aligned} \frac{\partial B_r}{\partial t} &= -R_\alpha B_\varphi - \frac{\pi^2}{4} B_r + \lambda^2 \left(\frac{\partial}{\partial r} \left(\frac{\partial}{r \partial r} (r B_r) \right) + \frac{1}{r^2} \frac{\partial^2 B_r}{\partial \varphi^2} - \frac{2}{r^2} \frac{\partial B_\varphi}{\partial \varphi} \right); \\ \frac{\partial B_\varphi}{\partial t} &= -R_\omega B_r - \frac{\pi^2}{4} B_\varphi + \lambda^2 \left(\frac{\partial}{\partial r} \left(\frac{\partial}{r \partial r} (r B_\varphi) \right) + \frac{1}{r^2} \frac{\partial^2 B_\varphi}{\partial \varphi^2} - \frac{2}{r^2} \frac{\partial B_r}{\partial \varphi} \right), \end{aligned} \tag{1}$$

where B_r and B_φ are the magnetic field components in the disc plane, R_α is the dimensionless amplitude of the alpha-effect, R_ω is the dimensionless amplitude of differential rotation, $\lambda = h/R$ is the disc aspect ratio, where h is the half-thickness of the galaxy disc, R is its radius. The distances are measured in galactic radii ($0 < r < 1$), and the time is measured in h^2/η, where η is the coefficient of turbulent diffusivity. A conventional estimate is $R_\alpha \sim 1$, $R_\omega \sim 10$.

The magnetic field growth is determined by the transition of the kinetic energy of interstellar motions to the magnetic field energy. So, the magnetic field cannot grow more than the equipartition value B^* that is described by Eq. [4]

$$\frac{B^{*2}}{8\pi} = \frac{\rho v^2}{2},$$

where ρ is the density of the interstellar gas and v is the velocity of turbulent motions. So the equipartition value is $B^* = v\sqrt{4\pi\rho}$.

Taking into account the saturation of the magnetic field growth, we can change the coefficient R_α by a nonlinear modification [14]

$$R_\alpha = \frac{R_{\alpha 0}}{1 + \left(B_r^2 + B_\varphi^2\right)/B^{*2}}.$$

If we assume $B^* = 1$ (this can be done by taking special dimensionless units for the magnetic field), Eq. (1) turns out:

$$\begin{aligned} \frac{\partial B_r}{\partial t} &= -\frac{R_\alpha}{1 + B_r^2 + B_\varphi^2} B_\varphi - \frac{\pi^2}{4} B_r + \\ &+ \lambda^2 \left(\frac{\partial}{\partial r} \left(\frac{\partial}{r \partial r} (r B_r) \right) + \frac{1}{r^2} \frac{\partial^2 B_r}{\partial \varphi^2} - \frac{2}{r^2} \frac{\partial B_\varphi}{\partial \varphi} \right); \\ \frac{\partial B_\varphi}{\partial t} &= -R_\omega B_r - \frac{\pi^2}{4} B_\varphi + \lambda^2 \left(\frac{\partial}{\partial r} \left(\frac{\partial}{r \partial r} (r B_\varphi) \right) + \frac{1}{r^2} \frac{\partial^2 B_\varphi}{\partial \varphi^2} - \frac{2}{r^2} \frac{\partial B_r}{\partial \varphi} \right). \end{aligned} \tag{2}$$

We assume that R_α is described by a random law:

$$R_\alpha = \begin{cases} 0.1 \text{ with probability } p; \\ 1 \text{ with probability } (1-p). \end{cases} \tag{3}$$

We assume that R_α is a piece constant function: for small time intervals $\Delta t \sim 10^{-2}$ it is constant, and after that it renews according to the random law. Sometimes we took some continuous distribution for R_α with thin peaks at 0.1 and 10. For the partial differential equations, we also assume that for every ring $n\Delta r < r < (n+1)\Delta r$ the coefficient R_α has a different value [15].

Of course, some effects that take place in galaxies with rapid processes must be described by more complicated parameterizations, but the main features can be described even by our quite simple model.

2. Local approach. At first we present the results for the simplest case of an infinitely thin disc and neglect the losses due to diffusion in the disc plane. Then the dynamo equations will be the following:

$$\begin{aligned} \frac{dB_r}{dt} &= -R_\alpha B_\varphi - \frac{\pi^2}{4} B_r; \\ \frac{dB_\varphi}{dt} &= -R_\omega B_r - \frac{\pi^2}{4} B_\varphi. \end{aligned} \tag{4}$$

If the parameters are deterministic, the solution will describe the exponential growth: $B_{r,\varphi} \sim \exp(\gamma t)$, where $\gamma = -\frac{\pi^2}{4} \pm \sqrt{D}$. $D = R_\alpha R_\omega$ is the so-called dynamo number. If $D \gtrsim 7$, the magnetic field will grow, else it will decay. $D_{cr} \approx 7$ is the critical value of the dynamo number that describes the character of the magnetic field evolution.

If we introduce a vector $\mathbf{B} = (B_r, B_\varphi)$, the dynamo equations (4) can be rewritten in the matrix form:

$$\frac{d}{dt}(B_r, B_\varphi) = (B_r, B_\varphi) \begin{pmatrix} -\frac{\pi^2}{4} & -R_\omega \\ -R_\alpha & -\frac{\pi^2}{4}, \end{pmatrix}$$

which can be solved as

$$\mathbf{B}(n\Delta t) = \mathbf{B}((n-1)\Delta t) \exp\left\{ -\frac{\pi^2}{4} \Delta t \right\} C_n,$$

where C_n is the transition matrix:

$$C_n = \begin{pmatrix} \cosh\left(\sqrt{R_\alpha R_\omega}\Delta t\right) & -\sqrt{\frac{R_\omega}{R_\alpha}}\sinh\left(\sqrt{R_\alpha R_\omega}\Delta t\right) \\ -\sqrt{\frac{R_\alpha}{R_\omega}}\sinh\left(\sqrt{R_\alpha R_\omega}\Delta t\right) & \cosh\left(\sqrt{R_\alpha R_\omega}\Delta t\right) \end{pmatrix}.$$

So, we can introduce a matrix [12, 16]:

$$C(n) = C_1 C_2 ... C_n,$$

and the magnetic field for each time moment $t = n\Delta t$ will be described by the formula:

$$\mathbf{B}(n\Delta t) = \exp\left\{-\frac{\pi^2}{4}\Delta t\right\}\mathbf{B}(0)C(n)$$

.

Let

$$\tan\theta = \frac{B_\varphi}{B_r}$$

. The angle θ at each time $t = n\Delta t$ has a distribution $\pi_n(\theta)$ which can be described by the so-called transition probability density $p(\theta, \chi)$:

$$\pi_{n+1}(\theta) = \int_{-\pi/2}^{\pi/2} p(\theta, \chi)\pi_n(\chi)\mathrm{d}\chi.$$

For $n \to \infty$, the distribution function has a limit [12]:

$$\pi_n(\theta) \to \pi_\infty(\theta).$$

Then, the magnetic field growth rate is

$$\gamma = \frac{1}{\Delta t} < \ln\|\mathbf{w}C_n\| > -\frac{\pi^2}{4},$$

where $\mathbf{w} = (\cos\theta, \sin\theta)$ and θ has a limit distribution $\pi_\infty(\theta)$.

First of all, we investigate the dynamo equations numerically for various values of α (Fig. 1). The magnetic field grows with $p < 0.43$ and decays with $p > 0.43$.

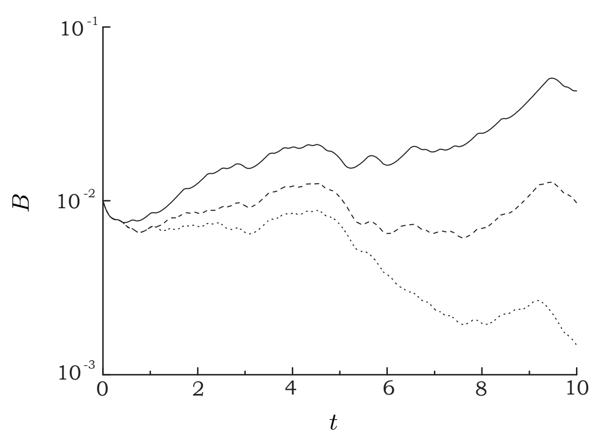

Fig. 1. Magnetic field growth for various p (linear local model). $R_\omega = 10$, R_α is described by Eq. (3). The solid line shows $p = 0.30$, the long-dashed one $p = 0.40$, the short-dashed one $p = 0.50$.

Table 1. Velocities of the exponential growth for different p with the local approach Eq. (4). $R_\omega = 10$, R_α is described by Eq. (3).

p	γ_B	$\gamma_{<B>}$	$\gamma_{<B^2>^{1/2}}$	$\gamma_{<B^3>^{1/3}}$	$\gamma_{<B^4>^{1/4}}$	γ_{theor}
0.30	0.224	0.235	0.238	0.241	0.244	0.250
0.40	0.056	0.062	0.067	0.070	0.074	0.065
0.50	–0.130	–0.122	–0.117	–0.113	–0.108	–0.116

Table 2. Velocities of the exponential growth for different p in the r-dependent system (6). $R_\omega = 10$, R_α is described by Eq. (3).

p	γ_B	$\gamma_{<B>}$	$\gamma_{<B^2>^{1/2}}$	$\gamma_{<B^3>^{1/3}}$	$\gamma_{<B^4>^{1/4}}$
0.30	0.221	0.229	0.230	0.232	0.233
0.40	0.050	0.058	0.059	0.061	0.062
0.50	–0.136	–0.127	–0.125	–0.123	–0.122

The typical growth rates of various statistical momenta of $B = \sqrt{B_r^2 + B_\varphi^2}$ are listed in Table 1. The higher momenta grow faster than the lower ones. It is called the intermittency effect. Then, we find the limiting probability density π (Fig. 2) and calculate the magnetic field growth rate analytically using this density. Numerical and analytical estimates are compared in Table 1.

We use afterward a nonlinear modification of system (4):

$$\begin{aligned} \frac{dB_r}{dt} &= -\frac{R_\alpha}{1 + B_r^2 + B_\varphi^2} B_\varphi - \frac{\pi^2}{4} B_r; \\ \frac{dB_\varphi}{dt} &= -R_\omega B_r - \frac{\pi^2}{4} B_\varphi. \end{aligned} \tag{5}$$

The results are illustrated in Fig. 3. It is seen that the field growth for small values of $B \ll 1$ is quite similar to the linear case (4). But after the growth saturates, the magnetic field does not stabilize as for the deterministic case (solid line). It oscillates near the equipartition value.

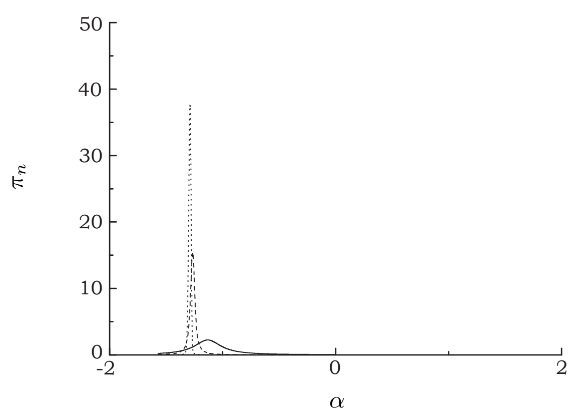

Fig. 2. The probability density for different n: solid line $n = 20$, long-dashed line $n = 50$, short-dashed line $n = 100$. $R_\omega = 10$, R_α is described by Eq. (3).

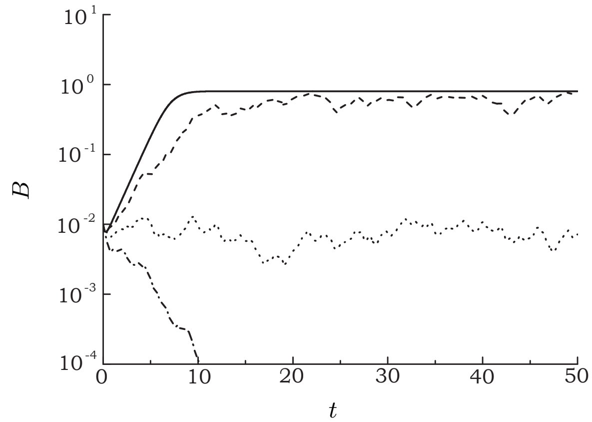

Fig. 3. Magnetic field growth for various p (non-linear local model). $R_\omega = 10$, R_α is described by Eq. (3). The solid line shows $p = 0.00$, the long-dashed line $p = 0.20$, the short-dashed line $p = 0.40$, the dot-dashed line $p = 0.60$.

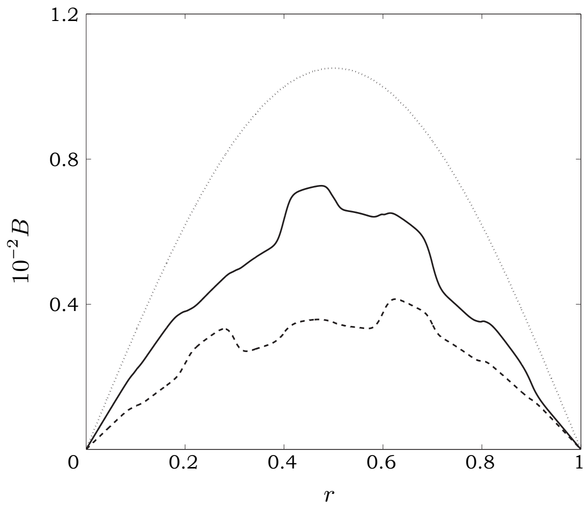

Fig. 4. The magnetic field at $t = 100\Delta t$ for different p (linear r-dependent model). The short-dashed line $p = 0.00$, the solid line $p = 0.30$, the long-dashed line $p = 0.60$. $R_\omega = 10$, R_α is described by Eq. (3).

3. Partial differential equations. To take into account some local effects, we also used a model with r-dependence [14]:

$$\frac{\partial B_r}{\partial t} = -\frac{R_\alpha}{1 + B_r^2 + B_\varphi^2} B_\varphi - \frac{\pi^2}{4} B_r + \lambda^2 \left(\frac{\partial}{\partial r} \left(\frac{\partial}{r \partial r} (r B_r) \right) \right);$$
$$\frac{\partial B_\varphi}{\partial t} = -R_\omega B_r - \frac{\pi^2}{4} B_\varphi + \lambda^2 \left(\frac{\partial}{\partial r} \left(\frac{\partial}{r \partial r} (r B_\varphi) \right) \right). \quad (6)$$

A typical dependence of B on r is shown in Fig. 4. The time dependence is quite similar to the local approach. The velocities of magnetic field growth are listed in Table 2. It can be seen that they are a bit smaller, which can be explained by the energy dissipation caused by a Laplace operator.

4. Conclusions. We have calculated the velocities of the magnetic field growth in the galaxy dynamo model with random coefficients both numerically and using the invariant measure technique. It is shown that there is an intermittency effect in this model: the higher momenta of the field grow faster than the lower ones [11]. The magnetic field evolution is qualitatively similar to the local model and to the r-dependent one. Some differences are assosiated with the field dissipation caused by the Laplace operator.

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