MARGINAL CONDITION FOR SPONTANEOUS OSCILLATIONS OF A THERMOACOUSTIC ENGINE COUPLED WITH A PIEZOELECTRIC ELEMENT. ANALYTICAL AND EXPERIMENTAL STUDY

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Abstract: This paper deals with the problem to derive a marginal condition for the onset of spontaneous thermoacoustic oscillations of a gas in a circular tube, subject to a variable shape of the temperature gradient along the side wall, with one end rigidly closed and the other closed by a piezoelectric element converter. In this study the acoustic impedance of the piezo element is arbitrary in order to achieve marginal conditions between those exhibited with rigidly closed end, and those with end opened onto free atmosphere. Moreover, marginal condition is outlined adopting a variable shape of the temperature gradient with respect to the position of the stack along the tube. The solution includes all dissipative effects related to the compressive and shear viscosity and the heat transmission in the boundary layer at the side wall and end wall.

1. Introduction.

This study investigates one of the promising candidates in the field of energy conversion, namely a standing waves thermoacoustic engine. In thermoacoustic systems heat is converted into acoustic energy and vice versa.

Nomenclature

$p_e = p_0$ value of the uniform pressure related to the equilibrium state, [Pa];
T_e local gas temperature in equilibrium conditions, [°C];
ρ_e local gas density in equilibrium conditions, related to the local temperature T_e, [kg/m^3];
ν_e kinematic viscosity, related to the local temperature T_e, [m^2/s];
a_e local sound speed in equilibrium conditions, related to the local temperature T_e, [m/s];
c_v specific heat capacity at constant volume of working fluid, [kJ/kg°C];
c_p specific heat capacity at constant pressure of working fluid, [kJ/kg°C];
T_H temperature at the hot, closed end, [°C];
T_0 temperature at the cold, open or piezo system mounted end, [°C]; R radius of the tube [m];
γ c_p/c_v; Pr = Prandtl number; $C = 1 + \frac{\gamma - 1}{\sqrt{Pr}}$; $C_T = {}^1/_2 + \frac{1}{\sqrt{Pr}}$ +Pr;
ω = angular frequency, [1/s] $\sigma = \frac{\omega L_T}{a_0}$ dimensionless angular frequency;
u_x velocity in the axial direction, [m/s]; u_r = velocity in the radial direction, [m/s];
$\delta_e = \frac{1}{R}\sqrt{\frac{\nu_e}{i\omega}}$, represent a measure of how is deep the boundary layer respect to the radius R
δ_0 δ_e calculated for $T = T_0$ at the low end wall temperature and C is a gas constant.
$b = C\,\delta_0$ frequency equation is obtained in the zero and in the first order by expanding the wave perturbed pressure equation with respect to b ; i = imaginary part;
$x_p \in [-L_{-}, +L_p]$ ("p" region) domain were the temperature gradient is negative and characterized by a slow slope. Quantities are designated by attaching a subscript "p" (e.g. T_{ep}, T'_p, $\tilde{T}_p$);

L_p abscissa at the end wall where is placed the piezo-system converter. The temperature in this point is the lowest one and is indicated by T_0;

$x_n \in [-L_n, -L_*]$ ("n" region) domain were the temperature gradient is negative and characterized by an high slope. Quantities are designated by attaching a subscript "n" (e.g. T_{en}, T'_n, $\tilde{T}_n$);

$-L_n$ abscissa value that localize the position of the end wall resonator respect to the origin of the system reference. The temperature in this point is the highest one and is indicated by T_H;

$L_T = L_n + L_p$ length of the tube, [m];

$-L_*$ abscissa value that localize the point where the slope of the temperature gradient changes dramatically his value and $T_e(-L_*)$ is equal to T_*;

λ_p is a parameter that governs the magnitude and the sign of the slope of the temperature gradient in the "p" region so that $\frac{a_{ep}}{T_{ep}}\frac{dT_{ep}}{dx_p} = 2\frac{a_0\lambda_p}{L_p}$;

λ_n is a parameter that governs the magnitude and the sign of the slope of the temperature gradient in the "n" region so that $\frac{a_{en}}{T_{en}}\frac{dT_{en}}{dx_n} = 2\frac{a_0\lambda_n}{L_n}$;

$U_p = i\frac{2C_T}{C}be^{-\frac{\lambda_p \xi_p}{L_p}}\lambda_p\left(\frac{L_T}{L_p\sigma}\right)^2\psi_p$; $I_p = 1 + i\frac{2C_T}{C}be^{-\frac{\lambda_p \xi_p}{L_p}}\lambda_p\left(\frac{L_T}{L_p\sigma}\right)^2\frac{i\lambda_p}{2}$;

$g = (1-(1+\lambda_n)b)(1+\lambda_n)$; $d = \frac{2C_T}{C}\lambda_n(1+\lambda_n)b$; $J = \frac{kr\cdot\sigma^2}{L_n\cdot(1+\lambda_n)^2(1-2\cdot(1+\lambda_n))\cdot\frac{b}{C}}$ where

kr is a parameter that governs the volume size of the resonator cavity

A configuration of the system under study is shown in figure 1

Figure 1: Schematic of standing-wave thermoacoustic engine integrated with a piezoelectric membrane (TAP).

The right theoretical frame in order to get the marginal condition was derived by Rott [1] [2] [3] and afterwards it was completed by Wheatley [4] and Swift [5]. Rott's works are about quarter wave length tube. The linearized problem requires to solve an eigenvalue problem for a second-order differential equation with variable coefficients in terms of the excess pressure wave. For smooth temperature distributions, this is a formidable task. Indeed Rott gave up the smooth profile of the temperature and adopted a discontinuous trend, thereby imposing a drastic discontinuity, though it is rather difficult to achieve experimentally. In 2001, the excellent work of Sugimoto et al [6,7], shed some light on what happens on the onset of thermoacoustic oscillations. Thanks to his work was discovered the mechanism whereby the boundary layer, under an appropriate temperature gradient, is able to supply a work to the wave pressure propagation up to exceed dissipative effects of the viscous boundary layer.

In what follows, this work adopts the Sugimoto's approach with only two minor extensions with the aim to offer two points of generalization. The first one is related to the boundary condition at one end, because the complex value of the piezoelectric impedance is taken into account, and the second effort is about the possibility to build a variable shape of temperature gradient by means of a sequence of piecewise parabolic distribution as shown in figure 2.

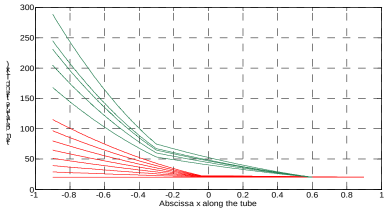

Figure 2: Some examples of the shape of the temperature imposed along the tube in stationary conditions.

2. A problem with a boundary layer structure: the main lossless flow and the viscous – thermal boundary layer

Here, we assume that the field of the acoustic flow has a boundary layer structure, which means that the influence of viscosity is confined to a thin layer near the wall. The flow is basically divided in a boundary layer region and the flow outside of it, namely the main flow.

By using the two system of equations regarding the main flow and the boundary layer on the side wall, it is possible to write down equation (1); that is a second order differential equation with variable coefficients of the axial coordinate "x" involving the Fourier transform "P" of the excess pressure p', in the main-flow region.

$$(1-2C\delta_e)a_e^2\frac{d^2P}{dx^2}+[1-2(C+C_T)\delta_e]\frac{a_e^2}{T_e}\frac{dT_e}{dx}\frac{dP}{dx}+\omega^2P=0 \qquad (1)$$

Equation (1) has been proposed by Sugimoto, (see eq. 23 in [7]) and it was used to get the stability analysis for the marginal conditions.

3. Boundary conditions at the ends wall of the tube.

When the tube is rigidly closed at one end and open on the other side, the boundary conditions for the main flow are well known, namely:

$\left(\frac{dp'}{dx}=-\frac{\frac{\gamma-1}{\sqrt{Pr\,\sqrt{\nu_L}}}}{a_L^2}\frac{d^{3/2}p'}{dt^{3/2}}\right)_{x=0}$ at the closed end; $(p'=0)_{x=L_p}$ at the opened end.

By using the idea of a renormalization of eq. (1) and in the framework of the first-order theory of the boundary layer, the frequency equation is then derived from the boundary conditions at the both ends of the tube when the temperature distribution is parabolic, from which the marginal condition, eq. (2), is obtained in closed form, in terms of the dimensionless angular frequency, $\sigma=\frac{\omega L}{a_0}$, as a function of the ratio, T_H/T_0;

$$i\psi\left(\frac{e^{iK^+\xi_L}+e^{iK^-\xi_L}}{e^{iK^+\xi_L}-e^{iK^-\xi_L}}\right)=\frac{\beta}{2}-\frac{2C_T}{C}\beta b+\left(1-\frac{1}{C}\right)\frac{R}{L}\sigma^2 b \qquad (2)$$

where β is a parameter that sets the slope of the parabola: $\frac{T_e(x)}{T_0}=\left(1+\beta\frac{x}{L}\right)^2$; K^+, K^- are the wave-numbers and they are given by: $K^{\pm}L=-\frac{i}{2}\beta\pm\psi$ with $\psi=\sqrt{\sigma^2-\frac{\beta^2}{4}}$. The solid curve in figure 3 shows the frequency of the neutral oscillations in the lossless case, as the magnitude of the thermoviscous effects increases, the value of sigma and temperature ratio decreases along the solid curve, but tends to deviate from the lossless trend

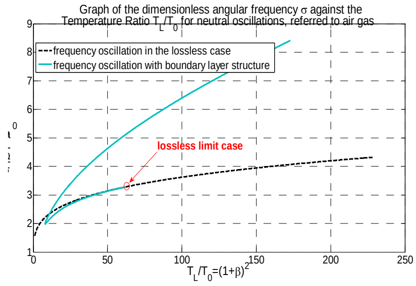

Figure 3: Sugimoto's case: marginal curve when a parabolic temperature gradient is imposed along the side wall tube. One end is opened and the other is rigidly closed. The curve represented in dashed lines is adopted for lossless case, whereas continuous is referred when loss are taken into account.

4. New boundary conditions

The effort of this work is to get new boundary conditions when the tube is rigidly closed at one end and at the other side is closed by means of a piezoelectric element with a variable impedance and the side wall is subject to a variable shape of the temperature. Eqs. (3) and (4), in the frequency domain, respectively represent the new boundary condition in $x = L_p$, where piezoelectric element is placed and the boundary condition in $x = 0$ where end wall is rigidly closed.

$$\left(\frac{dP}{dx} = -\frac{1 + G\left(Z_{piezo}\right)^{-1}}{G} i\omega Q_L P\right)_{x=L_p} \qquad (3)$$

where $\frac{1}{G} = \frac{\frac{\gamma - 1}{\sqrt{Pr}\,\square\,1}}{a_0^2}\sqrt{i\omega\nu_L 1\omega P}$ and $Z_{piezo} = \frac{\mathcal{F}\{p'\}}{\mathcal{F}\{u'_x\}}$ is the impedance of the piezoelectric system.

$$\left(\frac{dP}{dx} = -\frac{\frac{\gamma - 1}{\sqrt{Pr\,\square\,\sqrt{\nu_L}}}}{a_0^2}\sqrt[2]{i\omega^3}\,P\right)_{x=0} \qquad (4)$$

When Z_{piezo} goes to ∞ (referred to the physical condition where end wall is rigidly closed), the new boundary condition converges just to the previous one, in accordance with Sugimoto et al. [7].

$\log \square \eta_{\downarrow}(L_{\downarrow}n)/(\eta_{\downarrow}p^{\uparrow}*)\square + b(\eta_{\downarrow}(L_{\downarrow}n) - \eta_{\downarrow}p^{\uparrow}*) \times$ "Λ" ↓"n" /("4" "ψ" ↓"n") "log" □{[("Λ" ↓"n" /("2" "L" ↓"n")
(5)

Quantities in equation (5) not defined yet, are explained in appendix A. A set of marginal curves in the loss case, as a function of the piezo-impedance, and the shape of the temperature trend (in terms of L_* and L_p) is depicted in figures 4, 5 and 6.

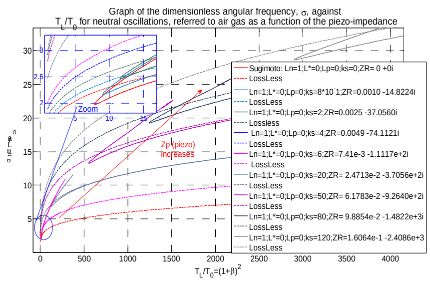

Figure 4. Marginal curve when a parabolic temperature gradient is imposed along the side wall tube. One end is opened and the other is closed by a piezo-electric element. The acoustic impedance of the piezo-element is arbitrary in order to achieve marginal conditions between those exhibited with rigidly closed end, and those with end opened to free atmosphere.

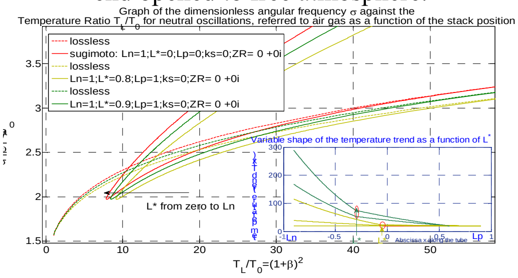

Figure 5. Marginal curve when the temperature trend is stretched due to different position of the stack, (L_* is different from zero). The curve represented in dashed lines is adopted for lossless case, whereas continuous is referred when loss are taken into account. One end is opened and the other is closed by a piezo-electric element with its impedance equal to zero.

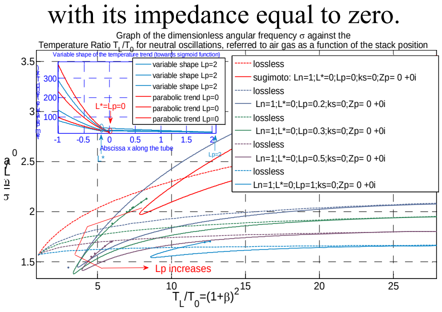

Figure 6 Marginal curve when the temperature trend is stretched due to different length of the tube respect to the position of the stack, (L_p is different from zero). The curve represented by dashed lines is for the lossless case, whereas the continuous line is referred when losses are taken into account. One end is opened and the other is closed by a piezo-electric element with its impedance equal to zero.

1. Conclusion

In this work, the main improvements to the best of our knowledge are summarized below. We found an analytical solution for the marginal conditions as a function of the value of the impedance of the piezoelectric element placed at the end wall of the tube. This solution is not only limited to the boundary conditions of opened and rigidly closed end.

The shape of the temperature gradient along the axial direction of the tube is variable and it is possible to realize changes where the slope is able to approximate at best the real temperature trends, as a function of the position of the stack along the tube and its length.

Based on the shape of the temperature gradient and the impedance of the piezoelectric element it is possible to determine the minimum threshold value for the temperature gradient required for the onset of oscillations.

Thanks to the flexibility of our model it is possible to get a theoretical prediction in order to match the resonant frequencies with the temperature ratio as a function of the electric load.

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Appendix A

Hereafter are listed all quantities that appear explicitly in eq. (5).

$$\eta_p^* = \eta_p(-L_*) = 1 + \lambda_p = \sqrt{\frac{T_*}{T_0}}; \qquad \eta_{L_n} = \eta_n(-L_n) = \sqrt{\frac{T_*}{T_0}} - \frac{\lambda_n}{L_n}(-L_n + L_*) = \sqrt{\frac{T_H}{T_0}};$$

$$\xi_p^* = -\frac{L_p}{\lambda_p}(\log\eta_p - b\,\lambda_p);$$

$$h = i \cdot e^{-\frac{C}{\sigma R}\sqrt{\frac{v_0 L_T}{2a_0}}\lambda_p} \cdot e^{i\frac{C}{\sigma R}\sqrt{\frac{v_0 L_T}{2a_0}}\lambda_p} \cdot \frac{2C_T}{C} b\lambda_n \left(\frac{L_T}{L_n\sigma}\right)^2; \quad R_n = 1 + i\frac{2C_T}{C}\lambda_n \frac{e^{-\frac{\lambda_n}{L_n}\xi_n^*}}{L_n K_n};$$

$$\psi_p = \sqrt{\left(\frac{\lambda_p}{2}\right)^2 - \left(\frac{L_p\sigma}{L_T}\right)^2};$$

$$..0020K_p = \frac{1}{L_p}\left(-i\frac{\lambda_p}{2} + \psi_p\right);$$

$$I_p = \frac{1}{L_p}\left(-i\frac{\lambda_p}{2} - \psi_p\right); \quad G_n = 1 + i\frac{2C_T}{C}\lambda_n \frac{e^{-\frac{\lambda_n}{L_n}\xi_n^*}}{L_n I_n}\psi_p; \quad W = \frac{1 - g \cdot d \cdot J \cdot (L_T)^2}{L_n \cdot \sigma^2};$$

$$X_{n,p} = i\frac{\lambda_{n,p}}{L_{n,}\mathrm{p}}; \qquad Y_{n,p} = \frac{\psi_{n,p}}{L_{n,p}}; \quad S = \frac{\mathrm{L_{po}}\varrho_0\omega}{Z_{ptezo}}$$

$$B = -\left(\frac{U_p}{R_n} + \frac{G_n L_n}{2\psi_n}\left(Y_p - \frac{U_p(X_n + Y_n)}{R_n}\right)\right);$$

$$P = \frac{U_p}{R_n} - L_n h\left(Y_p - \frac{U_p(X_n + Y_n)}{R_n}\right);$$

$$A = \frac{I_p}{R_n} - \frac{G_n L_n}{2\psi_n}\left(X_p - \frac{I_p(X_n + Y_n)}{R_n}\right); \quad D = \frac{I_p}{R_n} - L_n h\left(X_p - \frac{I_p(X_n + Y_n)}{R_n}\right);$$

$$V_s = \frac{\lambda_p}{2} + S - S \cdot b + \mathrm{L}_p \cdot T\frac{\sigma^2}{L_T}b; \qquad \psi_p = V_s - \frac{1}{2} \cdot S \cdot b \cdot (1 - b) \cdot \frac{2C_T}{C} \cdot \left(\frac{\lambda_p L_T}{\mathrm{L}_p \sigma}\right)^2$$