MOTION OF AN INSULATING SOLID PARTICLE NEAR A PLANE BOUNDARY UNDER THE ACTION OF UNIFORM AMBIENT ELECTRIC AND MAGNETIC FIELDS

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Abstract: This work presents a boundary approach to accurately compute at a reasonable cpu time cost the rigid-body motion of a solid and insulating particle immersed above a plane wall in a conducting liquid subject to ambient uniform electric and magnetic fields. Both insulating or perfectly conducting walls are addressed and the advocated formulation holds for arbitrary-shaped and arbitrary-located particles. It reduces to the determination of a few surface quantities on the particle boundary by numerically inverting seven boundary-integral equations.

1. Introduction

It is known, both theoretically [1] and experimentally [2], that an insulating solid particle suspended in a Newtonian and conducting liquid with uniform viscosity μ and conductivity $\sigma > 0$ migrates when subject to uniform ambient electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$. The particle rigid-body motion has translational velocity $\mathbf{U}$ (the velocity of one point attached to the particle) and angular velocity $\mathbf{\Omega}$ depending upon (σ, μ), the particle's geometry and the fields $\mathbf{E}$ and $\mathbf{B}$. For example, an insulating sphere with radius a translates without rotating at the velocity $\mathbf{U} = -a^2\sigma[\mathbf{E} \wedge \mathbf{B}]/(6\mu)$ (see [1]) whereas non-spherical insulating particles in general both rotate and translate [3,4].

In applications the liquid is however bounded and the particle's motion prevailing in an unbounded liquid domain might be strongly affected by the particle-boundary interactions. Such interactions have been investigated in the literature [5-7] by addressing the case of a particle located near a plane, solid and motionless wall Σ and distinguishing two different cases: the Case 1 of an insulating plane wall Σ in which the applied uniform electric field $\mathbf{E}$ is parallel to the wall and the Case 2 of an perfectly conducting plane wall Σ in which the uniform electric field $\mathbf{E}$ is normal to the wall. Case 1 has been handled in [7] but solely for a *distant and spherical* insulating particle while [5] copes only with Case 2 for a conducting or insulating and arbitrarily-located sphere. Finally, [6] deals with a non-spherical conducting particle in Case 2 by appealing to a boundary approach which has in practice been implemented solely when both vectors $\mathbf{E}$ are $\mathbf{B}$ are normal to the wall Σ. The present work proposes to solve the problem for an insulating particle with arbitrary shape and location in both Case 1 and Case 2 by exploiting a new and efficient flow decomposition.

2. Assumptions and governing electric and hydrodynamic problems

As sketched in fig 1, we consider a solid conducting particle $\mathcal{P}$ freely suspended in a Newtonian liquid metal, of uniform viscosity μ and conductivity $\sigma > 0$, above the $x_3 = 0$

plane and solid wall Σ. The particle has center of volume O' and smooth boundary S with unit normal $\mathbf{n}$ directed into the liquid domain Ω.

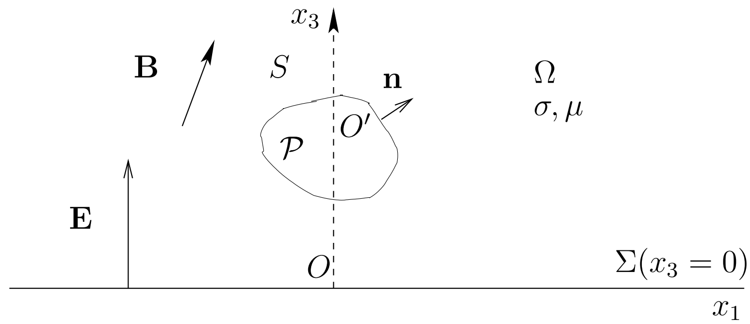

Figure 1: A solid and insulating particle freely suspended in a Newtonian liquid metal in the vicinity of the $x_3 = 0$ solid plane and motionless wall Σ. Here, the wall is perfectly conducting (Case 2) with the applied electric field $\mathbf{E}$ parallel with $\mathbf{e}_3$.

Under uniform ambient electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ a liquid flow driven by the Lorentz body-force takes place and induces, by viscosity, a rigid-body migration of the particle $\mathcal{P}$. With respect to a Cartesian system (O, x_1, x_2, x_3) attached to the wall, such a so-called electro-magneto-phoretic motion is described by the particle translational velocity $\mathbf{U}$ (here the velocity of its point O') and angular velocity $\mathbf{\Omega}$. This works presents a boundary approach to determine the particle motion $(\mathbf{U}, \mathbf{\Omega})$ whatever the particle shape and location for two quite different types of walls:

(i) Case 1: an insulating wall with a uniform applied electric field $\mathbf{E}$ parallel with the wall Σ.

(ii) Case 2: a perfectly conductting wall with, as illustrated in fig 1, a uniform applied electric field $\mathbf{E}$ normal to the wall Σ.

Note that [7] solely deals with Case 1 for a *distant and spherical* particle whereas [6] solely considers the Case 2 for a particle with arbitrary shape and location.

The insulating particle affects the ambient electric field and the disturbed electric field reads $\mathbf{E} - \nabla\phi$ in the liquid domain Ω. The function ϕ satisfies the well-posed problem

$$\nabla^2\phi = 0 \text{ in } \Omega, \ \nabla\phi \to \mathbf{0} \text{ as } r = |\mathbf{OM}| \to \infty, \tag{1}$$

$$\nabla\phi.\mathbf{n} = \mathbf{E}.\mathbf{n} \text{ on } S, \ \nabla\phi.\mathbf{e}_3 = 0 \text{ on } \Sigma \text{ in Case 1}, \phi = 0 \text{ on } \Sigma \text{ in Case 2}. \tag{2}$$

Here, (1)-(2) is efficiently solved by using for each Case i the following integral representation

$$\phi(\mathbf{x}) = \frac{1}{4\pi}\int_S q(\mathbf{y})\{\frac{1}{|\mathbf{x}-\mathbf{y}|} - (-1)^i\frac{1}{|\mathbf{x}-\mathbf{y}'|}\}dS(\mathbf{y}) \quad \text{for } \mathbf{x} \text{ in } \Omega \tag{3}$$

where $\mathbf{y}'$ designates the symmetric of the point $\mathbf{y}$ with respect to the wall Σ and the unknown surface charge density q on the particle surface S is obtained by enforcing the condition $\nabla\phi.\mathbf{n} = \mathbf{E}.\mathbf{n}$ on S (which results in a boundary-integral equation given in §3).

As previously mentioned, the liquid flows with pressure Q and velocity $\mathbf{u}$ with magnitude $V > 0$. The particle length scale a is such that the flow Reynolds number $\mathrm{Re} = \rho V a/\mu$ vanishes. Assuming also vanishing Hartmann and magnetic Reynolds numbers, the magnetic field $\mathbf{B}$ is not disturbed and $(\mathbf{u}, Q)$ becomes a quasi-steady Stokes flow driven by the non-uniform Lorentz body force $\mathbf{f} = \sigma(\mathbf{E} - \nabla\phi) \wedge \mathbf{B}$. Setting $Q = P + \sigma(\mathbf{E} \wedge \mathbf{B}).\mathbf{x}$, one arrives at the following key problem for the flow $(\mathbf{u}, P)$

$$\nabla.\mathbf{u} = 0 \text{ and } \mu\nabla^2\mathbf{u} = \nabla P + \sigma\nabla\phi\wedge\mathbf{B} \text{ in } \Omega, \tag{4}$$
$$\mathbf{u} = \mathbf{U} + \mathbf{\Omega}\wedge\mathbf{O'M} \text{ on } S,\ \mathbf{u} = \mathbf{0} \text{ on } \Sigma,\ (\mathbf{u}, P) \to (\mathbf{0}, 0) \text{ as } |\mathbf{x}| \to \infty. \tag{5}$$

The particle rigid-body motion $(\mathbf{U}, \mathbf{\Omega})$ occurring in (5) has to be determined by enforcing additional relations. For a particle with negligible inertia those conditions are obtained by requiring the particle to be force-free and torque-free. If the flow $(\mathbf{u}, P)$ has stress tensor $\boldsymbol{\sigma}$ one then arrives at, recalling that O' is the particle center of volume and setting $\mathbf{x}' = \mathbf{O'M}$,

$$\mathbf{F} := \int_S \boldsymbol{\sigma}.\mathbf{n}dS = \sigma\mathcal{V}_{\mathcal{P}}(\mathbf{E}\wedge\mathbf{B}),\ \ \mathbf{C} := \int_S \mathbf{x}'\wedge\boldsymbol{\sigma}.\mathbf{n}dS = \mathbf{0} \tag{6}$$

where $\mathcal{V}_{\mathcal{P}}$ designates the particle volume. At a very first glance, one has to solve (4)-(6) in order to gain the desired particle migration $(\mathbf{U}, \mathbf{\Omega})$. The next sections show how one can actually circumvent the determination of the flow $(\mathbf{u}, P)$ by resorting to a suitable boundary formulation which finally reduces to the determination of a few surface quantities on the particle surface S!

3. Flow decomposition and key boundary-integral equations

By linearity, it is useful to adopt the following decompositions $\mathbf{u} = \mathbf{u}_h + \mathbf{w} + \mathbf{v}$ and $P = p_h + p$ such that the flow $(\mathbf{u}_h, p_h)$ obeys (4)-(5) for $\sigma = 0$ while the other flows satisfy

$$\nabla.\mathbf{w} = 0 \text{ and } \mu\nabla^2\mathbf{w} = \sigma\nabla\phi\wedge\mathbf{B} \text{ in } \Omega, \mathbf{w} \to \mathbf{0} \text{ as } |\mathbf{x}| \to \infty, \tag{7}$$
$$\nabla.\mathbf{v} = 0 \text{ and } \mu\nabla^2\mathbf{v} = \nabla p \text{ in } \Omega, \tag{8}$$
$$\mathbf{v} = -\mathbf{w} \text{ on } S,\ \mathbf{v} = -\mathbf{w} \text{ on } \Sigma,\ (\mathbf{v}, p) \to (\mathbf{0}, 0) \text{ as } |\mathbf{x}| \to \infty. \tag{9}$$

The flow $(\mathbf{u}_h, p_h)$ exerts on the moving particle a force $\mathbf{F}_h$ and a torque $\mathbf{C}_h$ (with respect to O') which are obtained by introducing six auxiliary Stokes flows $(\mathbf{u}_L^{(i)}, p_L^{(i)})$ (for $i = 1, 2, 3$ and $L = t, r$) free from body force, quiescent far from $\mathcal{P}$ and obeying the specific boundary conditions

$$\mathbf{u}_L^{(i)} = \mathbf{0} \text{ on } \Sigma,\ \mathbf{u}_t^{(i)} = \mathbf{e}_i \text{ on } S,\ \mathbf{u}_r^{(i)} = \mathbf{e}_i\wedge\mathbf{x}' \text{ on } S. \tag{10}$$

Upon introducing the surface tractions $\mathbf{f}_L^{(i)}$ exerted on S by the flows $(\mathbf{u}_L^{(i)}, p_L^{(i)})$ and the second-rank tensors $\mathbf{K}, \mathbf{W}, \mathbf{V}$ and $\mathbf{D}$ with Cartesian components

$$K_{ij} = -[\int_S \mathbf{e}_j.\mathbf{f}_t^{(i)}dS]/\mu, \quad W_{ij} = -[\int_S (\mathbf{e}_j\wedge\mathbf{x}').\mathbf{f}_r^{(i)}dS]/\mu, \tag{11}$$
$$V_{ij} = -[\int_S (\mathbf{e}_j\wedge\mathbf{x}').\mathbf{f}_t^{(i)}dS]/\mu, \quad D_{ij} = -[\int_S \mathbf{e}_j.\mathbf{f}_r^{(i)}dS]/\mu \tag{12}$$

one immediately gets the relations

$$\mathbf{F}_h = -\mu\{\mathbf{K}.\mathbf{U} + \mathbf{V}.\mathbf{\Omega}\},\ \ \mathbf{C}_h = -\mu\{\mathbf{D}.\mathbf{U} + \mathbf{W}.\mathbf{\Omega}\}. \tag{13}$$

The flow $\mathbf{w}$ has zero pressure and stress tensor $\boldsymbol{\sigma}_{\mathbf{w}}$ whereas the flow $(\mathbf{v}, p)$ has stress tensor $\boldsymbol{\sigma}_{\mathbf{v}}$. Such flows exert on the particle forces and torques (with respect to O') given by

$$\mathbf{F}_{\mathbf{w}} = \int_S \boldsymbol{\sigma}_{\mathbf{w}}.\mathbf{n}dS,\ \mathbf{C}_{\mathbf{w}} = \int_S \mathbf{x}'\wedge\boldsymbol{\sigma}_{\mathbf{w}}.\mathbf{n}dS,\ \mathbf{F}_{\mathbf{v}} = \int_S \boldsymbol{\sigma}_{\mathbf{v}}.\mathbf{n}dS,\ \mathbf{C}_{\mathbf{v}} = \int_S \mathbf{x}'\wedge\boldsymbol{\sigma}_{\mathbf{v}}.\mathbf{n}dS. \tag{14}$$

Accordingly, the relations (6) become the well-posed linear system

$$\mathbf{K}.\mathbf{U}+\mathbf{V}.\mathbf{\Omega}=\{\mathbf{F_w}+\mathbf{F_v}-\sigma\mathcal{V}_{\mathcal{P}}(\mathbf{E}\wedge\mathbf{B})\}/\mu,\ \mathbf{D}.\mathbf{U}+\mathbf{W}.\mathbf{\Omega}=\{\mathbf{C_w}+\mathbf{C_v}\}/\mu \qquad (15)$$

for the unknown particle rigid-body motion $(\mathbf{U},\mathbf{\Omega})$. This system is shown in this paper to be entirely determined from the knowledge of a very few surface quantities on the particle boundary S : the previously-introduced (see §2) surface charge density q and the tractions $\mathbf{f}_L^{(i)}$. Such quantities are found to obey the following boundary-integral equations

$$\frac{q(\mathbf{x})}{2}+\frac{1}{4\pi}\int_S\{\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3}-(-1)^i\frac{\mathbf{x}-\mathbf{y}'}{|\mathbf{x}-\mathbf{y}'|^3}\}.\mathbf{n}(\mathbf{x})q(\mathbf{y})dS(\mathbf{y})=-[\mathbf{E}.\mathbf{n}](\mathbf{x})\text{ for }\mathbf{x}\text{ on }S, \qquad (16)$$

$$-\frac{1}{8\pi\mu}\int_S G_{jk}(\mathbf{x},\mathbf{y})[\mathbf{f}_L^{(i)}.\mathbf{e}_k](\mathbf{y})dS(\mathbf{y})=[\mathbf{u}_L^{(i)}.\mathbf{e}_j](\mathbf{x})\ \text{ for }\mathbf{x}\text{ on }S \qquad (17)$$

where in (17) summation over indices k holds and $G_{jk}(\mathbf{x},\mathbf{y})$ denotes the Cartesian component of the so-called Green tensor analytically obtained in [8].

4. Relevant analytical solution for w and use of the reciprocal identity

As the reader may check using the representation (3), *one* solution to (7) is

$$\mathbf{w}(\mathbf{x})=\frac{\sigma}{8\pi\mu}\Big[\int_S q(\mathbf{y})\{\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}-(-1)^i\frac{\mathbf{x}-\mathbf{y}'}{|\mathbf{x}-\mathbf{y}'|}\}dS(\mathbf{y})\Big]\wedge\mathbf{B}\text{ for }\mathbf{x}\text{ in }\Omega\cup S\cup\Sigma. \qquad (18)$$

The associated surface traction $\boldsymbol{\sigma}_\mathbf{w}.\mathbf{n}$ on the particle surface S then reads

$$[\boldsymbol{\sigma}_\mathbf{w}.\mathbf{n}](\mathbf{x})=-\frac{\sigma}{8\pi}\int_S q(\mathbf{y})\Big[\frac{(\mathbf{x}-\mathbf{y}).\mathbf{n}(\mathbf{x})(\mathbf{x}-\mathbf{y})\wedge\mathbf{B}+\mathbf{n}(\mathbf{x}).[(\mathbf{x}-\mathbf{y})\wedge\mathbf{B}](\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3}$$
$$-(-1)^i\frac{(\mathbf{x}-\mathbf{y}').\mathbf{n}(\mathbf{x})(\mathbf{x}-\mathbf{y}')\wedge\mathbf{B}+\mathbf{n}(\mathbf{x}).[(\mathbf{x}-\mathbf{y}')\wedge\mathbf{B}](\mathbf{x}-\mathbf{y}')}{|\mathbf{x}-\mathbf{y}'|^3}\Big]dS(\mathbf{y}). \qquad (19)$$

Thus, one can evaluate the required force $\mathbf{F_w}$ and torque $\mathbf{C_w}$ from the knowledge of q. Furthermore, $(\mathbf{v},p)$ and the flow $(\mathbf{u}_L^{(i)},p_L^{(i)})$ with stress tensor $\boldsymbol{\sigma}_L^{(i)}$ are Stokes flows free from body force and quiescent far from the particle. Therefore, the usual reciprocal identity [9] applies and yields

$$\int_{S\cup\Sigma}\mathbf{v}.\boldsymbol{\sigma}_L^{(i)}.\mathbf{n}dS=\int_{S\cup\Sigma}\mathbf{u}_L^{(i)}.\boldsymbol{\sigma}_\mathbf{v}.\mathbf{n}dS. \qquad (20)$$

From the definition (14), the property (20) and the boundary conditions (9)-(10) one then gets the key relations

$$\mathbf{F_v}.\mathbf{e}_i=-\int_S\mathbf{w}.\mathbf{f}_t^{(i)}dS-\int_\Sigma\mathbf{w}.\boldsymbol{\sigma}_t^{(i)}.\mathbf{e}_3dS,\ \mathbf{C_v}.\mathbf{e}_i=-\int_S\mathbf{w}.\mathbf{f}_r^{(i)}dS-\int_\Sigma\mathbf{w}.\boldsymbol{\sigma}_r^{(i)}.\mathbf{e}_3dS. \qquad (21)$$

By virtue of (18), the velocity $\mathbf{w}$ required on the boundaries S and Σ when applying the above links (14) is gained from the charge density q on the particle surface S. Finally, inspecting (21) shows that one also needs to compute each stress tensors $\boldsymbol{\sigma}_L^{(i)}$ on the plane wall Σ. Again, this is achieved from the knowledge of the traction $\mathbf{f}_L^{(i)}$ on the surface S, using this time the key integral representation

$$\mathbf{e}_l.\boldsymbol{\sigma}_L^{(i)}(\mathbf{x}).\mathbf{e}_k=\frac{1}{8\pi}\int_S T_{lkj}(\mathbf{x},\mathbf{y})[\mathbf{f}_L^{(i)}.\mathbf{e}_k](\mathbf{y})dS(\mathbf{y})\quad\text{for }\mathbf{x}\text{ in }\Omega\cup\Sigma \qquad (22)$$

where the components T_{lkj} are available in closed analytical form in [10]. Those results are too long to be reproduced here. In exploiting (21) one then makes use of (22) on the wall Σ.

5. Concluding remarks

Owing to a suitable flow decomposition, it has been possible to reduce the determination of the particle rigid-body motion by solely appealing to a few surface quantities on the particle boundary: the charge density and the tractions exerted there when the particle either translates or rotates in absence of ambient electric and magnetic fields. The resulting boundary approach, valid whatever the particle shape and location, ends up with seven boundary-integral equations governing those key quantities. Such integral equations must be numerically solved in general or asymptotically inverted for a distant particle. Both circumstances will be addressed at the oral presentation which will report numerical results for spherical and non-spherical insulating particles and differents types (Case 1 and Case 2) of walls.

6. References

[1] Leenov, D.; Kolin, A.: Theory of electromagnetophoresis. I. Magnetohydrodynamic forces experienced by spherical and symmetrically oriented cylindrical particles. J. Chem. Phys. 22 (1954) 683-688.

[2] Marty, P.; Alemany, A.: Theoretical and experimental aspects of electromagnetic separation. In Metallurgical Applications of Magnetohydrodynalics (ed. H. K. Moffatt & M. R. E. Proctor). Metals Society. (1984) 245-259.

[3] Moffatt, H. K.; Sellier, A.: Migration of an insulating particle under the action of uniform ambient electric and magnetic fields. Part 1. General theory. J. Fluid Mech. 464 (2002) 279-286.

[4] Sellier, A.: Migration of an insulating particle under the action of uniform ambient electric and magnetic fields. Part 2. Boundary formulation and ellipsoidal particles. J. Fluid Mech. 488 (2003) 335-353.

[5] Sellier, A.: Migration of a solid conducting sphere immersed in a liquid metal near a plane conducting solid wall under the action of uniform ambient electric and magnetic fields. MagnetohydroDynamics. 42 Issue 2-3 (2006) 317-3262.

[6] Sellier, A.: Electromagnetic motion of a conducting particle immersed in a liquid metal near a solid wall. Proceedings of EPM2006, Sendai, 23-27 October 2006.

[7] Yariv, E & Miloh, T. Boundary effets on electro-magneto-phoresis. J. Fluid Mech. 622 (2009) 195-207.

[8] Blake, J. R.: A note on the image system for a Stokeslet in a no-slip boundary, Proc. Camb. Phil. Soc. 70 (1971) 303-310.

[9] Happel J.; Brenner H.: Low Reynolds number hydrodynamics, Martinus Nijhoff 1973.

[10] Pozrikidis C.: Boundary integral and singularity methods for linearized viscous flow, Cambridge University Press 1992.