Upwind Finite Difference Solution of MHD Pipe Flow in an Exterior Conducting Region using Shishkin and Bakhvalov Typed Layer Adapted Grids

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Abstract

We consider the magnetohydrodynamic (MHD) flow through a square pipe under the influence of a transverse magnetic field when the outside medium is also electrically conducting. MHD partial differential equations are of convection-diffusion type and it is well known that convection dominated problems have numerical instabilities on a uniform mesh with standard FDM. Therefore, an upwind FDM on Shishkin and Bakhvalov typed layer adapted grids are considered in order to obtain stable solutions for high values of the parameters. Results are visualized in terms of contour lines of the velocity and induced current.

1 Introduction

It is already known that there are many applications of MHD pipe flow such as the design of the cooling systems with liquid metals for nuclear reactors, electromagnetic pumps, MHD generators, and flowmeters measuring blood pressure, etc. The exact solution of the problem can be obtained only for some special cases [1, 2]. Therefore, there are many numerical methods applied to the solution of the MHD pipe flow (see [3, 4, 5] and references there in)

In this paper we consider MHD pipe flow of square cross-section under the influence of a transverse magnetic field when the outside medium is also electrically conducting. Governing coupled partial differential equations with coupled boundary conditions are obtained from the Navier-Stokes equations for conducting fluids, and Maxwells equations for electromagnetic field through Ohms law. The equations are written in non-dimensional form as [5];

(1)
$$\begin{aligned} \nabla^2 V(x,y) + ReRh\frac{\partial B}{\partial y}(x,y) &= -1 \\ \nabla^2 B(x,y) + Rm_1\frac{\partial V}{\partial y}(x,y) &= 0 \end{aligned} \qquad \text{in } \Omega_{in}$$

(2)
$$\nabla^2 B_{ext}(x,y) = 0 \qquad \text{in } \Omega_{ext}$$

with the no-slip condition on the pipe wall

(3)
$$V = 0 \qquad \text{on } \partial\Omega_{in} = \Gamma$$

and continuity conditions for the induced magnetic fields

(4)
$$B(x,y) = B_{ext}(x,y)\,, \qquad \text{on } \Gamma$$

(5)
$$\frac{1}{Rm_1}\frac{\partial B(x,y)}{\partial n} = \frac{1}{Rm_2}\frac{\partial B_{ext}(x,y)}{\partial n'}\,, \qquad \text{on } \Gamma$$

where n and n' are unit outward normals on Γ for the regions Ω_{in} and Ω_{ext}, respectively. Rm_1 and Rm_2 are the magnetic Reynolds numbers inside the pipe and in external medium.

We assume that cross section of the pipe is square and has sufficient length. The fluid is flowing through the pipe due to an applied constant pressure gradient $\frac{\partial p}{\partial z}$, and is viscous, incompressible, electrically conducting. The electrical permitivity and magnetic permeability of the fluid are assumed to be close to those of the external space. The axis of the pipe is coincident with the z-axis, and the y-axis is parallel with the magnetic induction at infinity. Thus, externally applied magnetic field with a constant intensity B_0 is assumed to be in y-direction. We also assume that the wall of the pipe and the outside medium are having the same electrical conductivity and magnetic permeability since the thickness of pipe wall is assumed to be very small (Fig 1). Although the external region is unbounded, an artificial boundary is considered far away from the pipe in order to perform numerical calculations. It is known that external induced magnetic field is almost zero at sufficiently far away distance from the pipe boundary. Therefore, on the artificial boundary Γ_∞, the external induced current (B_{ext}) is taken as either zero (homogenous Drichlet type) or free (homogenous Neumann type).

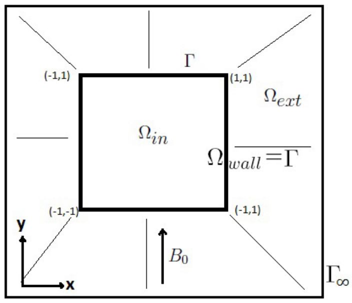


Figure 1: Problem definition

2 Mathematical Modelling

We have compared two different layer adapted meshes called Shishkin mesh and Bakhvalov mesh. However, in order to determine the structure of these meshes, we should transform the coupled equations to decoupled convection-diffusion typed equations as follows; Rewriting the equations by denoting $V_1 = V$ and $B_1 = \frac{ReRh}{M}B$, where $M = \sqrt{ReRhRm_1}$ is the Hartmann number of the fluid, the system (1) becomes [1]

$$\nabla^2 V_1 + M\frac{\partial B_1}{\partial y} = -1$$

(6) in Ω_{in}

$$\nabla^2 B_1 + M\frac{\partial V_1}{\partial y} = 0.$$

In order to decouple the equations, define new variables $U_1(x, y)$ and $U_2(x, y)$ as $U_1 = V_1 + B_1$ and $U_2 = V_1 - B_1$ which gives

$$\nabla^2 U_1 + M\frac{\partial U_1}{\partial y} = -1$$

(7)

$$\nabla^2 U_2 - M\frac{\partial U_2}{\partial y} = -1.$$

From this form of the equations, it is said that, the problem has a boundary layer in y-direction depending on the value of the Hartmann number M. Also it is well known that standard numerical methods for these type of equations are unstable for the large values of the convection coefficient and fail to give accurate results. Therefore, we should consider a modified mesh along y-direction. In this study, we will consider two different type of layer meshes.

a) Shishkin mesh

Shishkin mesh is a piecewise uniform mesh. Depending on the location of the boundary layer, the domain is divided into two section. The location of the transition point is chosen in a way that half of the discretization points are placed near the boundary which develops boundary layer .

If we assume that the domain is $[0, 1]$ and the boundary layer occurs at right hand side boundary (at the point $y = 1$), than the location of the transition point λ is calculated as follows [6];

$$\lambda = min(\frac{1}{2}, \frac{1}{M} \ln N) \tag{8}$$

where N is the total number of the division of the interval $[0, 1]$. Therefore, $\frac{N}{2}$ equally spaced points are on the interval $[0, 1 - \lambda]$ and the $\frac{N}{2}$ of them are again equally spaced on the interval $[1 - \lambda, 1]$. Explicitly,

$$y_i = \begin{cases} (1-\lambda)\,\frac{2i}{N} & i = 0, ..., N/2 \\ (1-2\lambda) + \frac{2\lambda i}{N} & i = N/2+1, ..., N. \end{cases} \tag{9}$$

b) Bakhvalov mesh

Bakhvalov mesh is defined as the modified version of Shishkin mesh. Similar to Shishkin mesh, the transition point is calculated as

$$\lambda = min(\frac{1}{2}, \frac{2}{M} \ln N). \tag{10}$$

Again $\frac{N}{2}$ equally spaced points lie on the interval $[0, 1 - \lambda]$. However, boundary layer part of the points are not equally spaced, they are distributed exponentially. Explicit formulation for the location of the points is given as [6];

$$y_i = \begin{cases} (1-\lambda)\,\frac{2i}{N} & i = 0, ..., N/2 \\ 1 + \frac{2}{M} \ln\left(\frac{N^2-2(N-i)(N-1)}{N^2}\right) & i = N/2+1, ..., N. \end{cases} \tag{11}$$

Since, our problem is symmetric with respect to x-axis, the boundary layers exist at both upper and lower walls of the pipe. Therefore, location of the adaptive mesh points are symmetric also with respect to x-axis.

The difference operators for the first and second order derives are defined as usually for non-uniform mesh. The continuity of the induced magnetic field and the relationship between the solution inside the pipe and the solution on the external region is satisfied with the coupled boundary conditions. An upwind discretized form is as;

$$\textbf{Top :}\ B_{i,j} = B^{ext}_{i,j} = \frac{m_{ty} B^{ext}_{i,j+1} + B_{i,j-1}}{1 + m_{ty}} \qquad \textbf{Bottom :}\ B_{i,j} = B^{ext}_{i,j} = \frac{B_{i,j+1} + m_{by} B^{ext}_{i,j-1}}{1 + m_{by}}$$

$$\textbf{Left :}\ B_{i,j} = B^{ext}_{i,j} = \frac{B_{i,j+1} + m_{lx} B^{ext}_{i,j-1}}{1 + m_{lx}} \qquad \textbf{Right :}\ B_{i,j} = B^{ext}_{i,j} = \frac{m_{rx} B^{ext}_{i+1,j} + B_{i-1,j}}{1 + m_{rx}}$$

where $m_{ty} = \frac{Rm_1(y_i - y_{i-1})}{Rm_2(y_{i+1} - y_i)}$, $m_{by} = \frac{Rm_1(y_{i+1} - y_i)}{Rm_2(y_i - y_{i-1})}$, $m_{lx} = \frac{Rm_1(x_{i+1} - x_i)}{Rm_2(x_i - x_{i-1})}$, $m_{rx} = \frac{Rm_1(x_i - x_{i-1})}{Rm_2(x_{i+1} - x_i)}$.

The induced magnetic field values at the corner points are assumed to be the average of the neighbouring points.

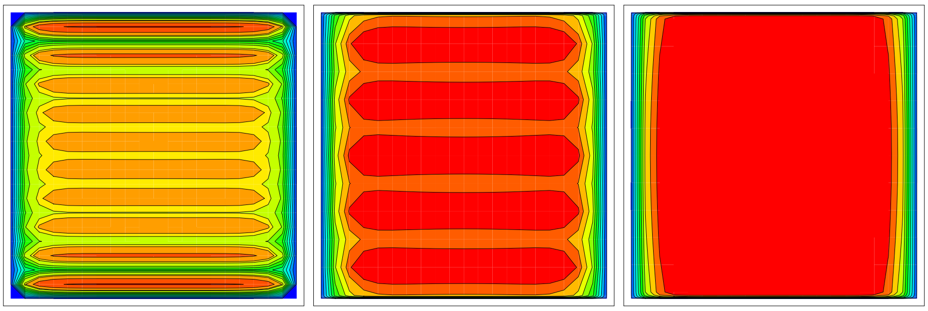
3 Numerical Results and Discussion

We consider a long pipe of square cross-section defined by $\{(x,y) : -1 \le x, y \le 1\}$. The artificial boundary Γ_∞ is assumed as $\{|x| = 3, -3 \le y \le 3 \bigcup |y| = 3, -3 \le x \le 3\}$. The pipe region is discretized by 21×21 mesh points both in x and y-directions. The behaviors of the velocity of the fluid and inside and outside induced currents (induced magnetic fields) are visualized in terms of contour plots for very high values of magnetic Reynolds numbers Rm_1, Reynolds number Re and magnetic pressure number Rh of the fluid.

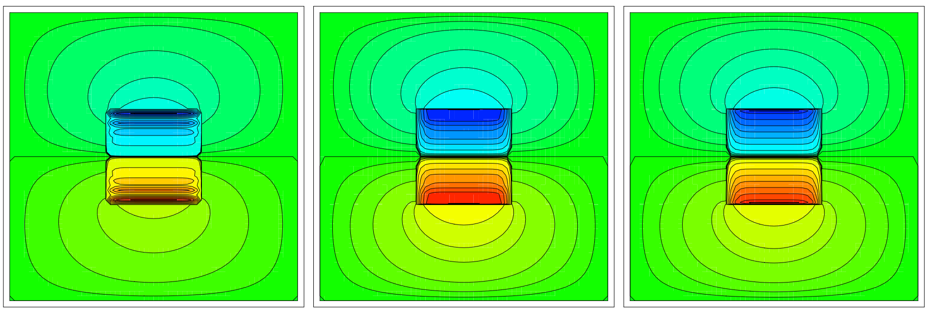
For moderate values of the Hartmann number ($Rm1 = 100, Rm2 = 1, Re = 1$ and $Rh = 10$), there is only slight disturbance on the velocity values obtained from uniform mesh. However, as Rm_1 getting large ($Rm_1 = 1000$), the need and effect of the stabilization are seen more clearly especially from velocity values. Unfortunately, stabilization with Shishkin typed mesh is not sufficient for stable solutions and Bakhvalov typed mesh is very effective compared to others. The numerical instabilities also start to appear in the induced magnetic field values obtained from uniform mesh (Figure 2).

Figure (3) shows equal velocity and induced current lines, respectively, for a very large value of Reynolds number $Re = 100$ when $Rm_1 = 100, Rm_2 = 1$ and $Rh = 10$. It seen from that, the effect of the stabilization in Bakhvalov types meshes is also seen from the induced current contours additional to the velocity contours. The similar behaviour is also seen from Figure (4) that displays the induced magnetic field contours for the case of Neumann type boundary condition on the artificial boundary.

Acknowledgements This work is supported by Karadeniz Technical University by Scientific Research Project (BAP) under Grant Number #10464.



(a) Velocity



(b) Induced currents

Figure 2: Uniform (left), Shishkin(center) and Bakhvalov(right) typed solution contours for $Rm_1 = 1000, Rm_2 = 1, Re = 1, Rh = 10$

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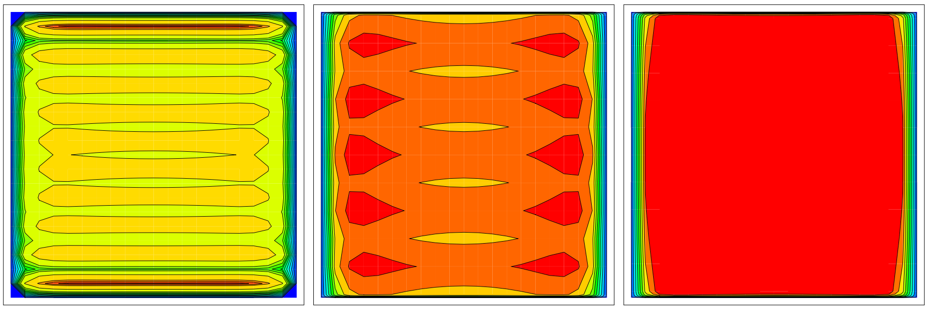
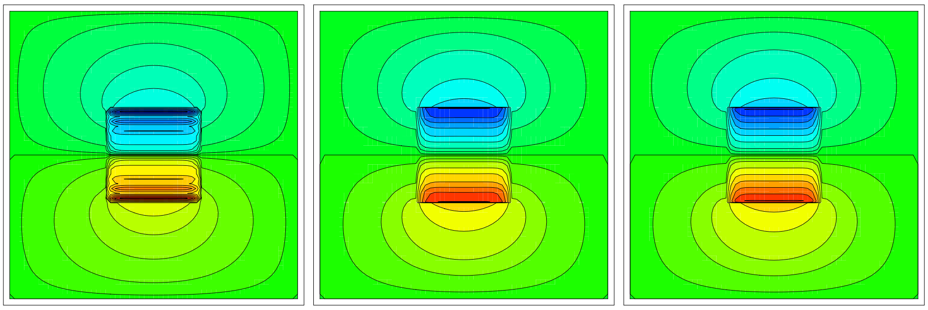


(a) Velocity



(b) Induced currents

Figure 3: Uniform (left), Shishkin(center) and Bakhvalov(right) typed solution contours for $Rm_1 = 100, Rm_2 = 1, Re = 100, Rh = 1$

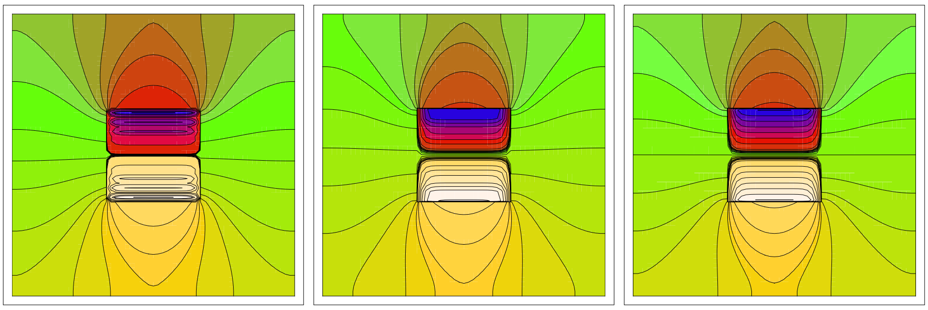


Figure 4: Uniform (left), Shishkin(center) and Bakhvalov(right) typed solution contours for $Rm_1 = 1000, Rm_2 = 1, Re = 1, Rh = 10$ and Neumann type boundary condition on the artificial boundary

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