Spectral Direct Numerical Simulations of low Rm MHD channel flows based on the least dissipative modes

KORNET Kacper and POTHERAT Alban

Applied Mathematics Research Centre, Coventry University

Abstract

We put forward a new type of spectral method for the direct numerical simulation of flows where anisotropy or very fine boundary layers are present. The mean idea is to take advantage of the fact that such structures are dissipative and that their presence should reduce the number of degrees of freedom of the flow. We applied the new method to calculate the evolution of freely decaying MHD turbulence between walls. Comparing our results with the cases calculated in a 3D periodic domain enables us to quantify the influence of the channel walls on the character of a freely decaying MHD turbulence.

1 Introduction

Simulations of liquid metal flows in channel and duct configurations under a strong magnetic field pose a difficult problem for existing numerical methods. The main obstacle is the linear increase in number of modes required to resolve thin Hartmann boundary layers with the intensity of the magnetic field B. Yet, the enormous dissipation incurred by friction and Joule dissipation in these layers decimates the degree of freedom of the flow when Ha becomes large. Their number can be estimated through the dimension d_M of the attractor of the underlying system, for which an upper bound was shown to scale as Ha^{-1} [4]. The fact that the number of modes needed to resolve the flow completely increases in numerical simulations at high Ha is therefore a property of the spectral method based on these polynomials, but does not reflect any physical constraint.
To overcome this problem we developed a new approach to the numerical calculations describing these flows. The solution of the flow is expressed in a base of eigenfunctions of the linear part of the governing equations and its adjoint. We show that in this approach the computational cost does not depend on the thickness of boundary layer and therefore it allows for performing calculations for high magnetic fields.

2 Governing equations

Flows of liquid metals in engineering applications are usually described within the frame of the Low Magnetic Reynolds number (Rm) approximation. This applies to problems where the flow is neither intense nor conductive enough to induce a magnetic field comparable to an externally

applied one. The full system of the induction equation and the Navier-Stokes equations for an incompressible fluid are then approximated to the first order in Rm, which represents the ratio of these two fields. This leads to the following system [6]:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu\Delta\mathbf{u} + \mathbf{j}\times\mathbf{B}\,, \tag{1}$$

$$\nabla\cdot\mathbf{u} = 0\,, \tag{2}$$

$$\nabla\cdot\mathbf{j} = 0\,, \tag{3}$$

$$\mathbf{j} = \sigma(-\nabla\Phi + \mathbf{u}\times\mathbf{B})\,, \tag{4}$$

where $\mathbf{u}$ denotes fluid velocity, $\mathbf{B}$ - magnetic field, $\mathbf{j}$ - electric current density, ν - kinematic viscosity, σ - electrical conductivity, Φ - electric potential. We consider a channel flow with a homogeneous transverse magnetic field $B\mathbf{e}_z$ and impermeable ($\mathbf{u}|_{wall} = \mathbf{0}$), electrically insulating ($\mathbf{j}\cdot\mathbf{n}|_{wall} = \mathbf{0}$) walls located at $z = \pm L/2$. In the xy directions we adopt the periodic boundary conditions with period L. Under this assumptions and using the reference scale L, time L^2/ν and velocity ν/L the above set of equations can be expressed in dimensionless form:

$$\frac{\partial \mathbf{u}}{\partial t} + P(\mathbf{u}\cdot\nabla)\mathbf{u} = \Delta\mathbf{u} - \frac{1}{Ha^2}\Delta^{-1}\partial_{zz}\mathbf{u}\,, \tag{5}$$

where $Ha = LB\sqrt{\sigma/\rho\nu}$ is the Hartman number and P denotes orthogonal projection onto the subspace of solenoidal fields.

3 Numerical methods

We express the solution of eq. (5) using a basis of eigenvectors of the operator $\mathcal{L}$ that represents its linear part. The features of flows at high Ha are strongly determined by the properties of this operator. Because of this, the set of modes built out of its eigenfunctions elements includes structures that are actually present in the flow. Laminar and turbulent Hartmann boundary layers that develop along the channel walls appear, in particular, as built-in features of these modes [2, 5]. They are therefore natural candidates to be used as elements of a functional basis in a numerical spectral scheme. Moreover, these modes all have negative eigenvalues, and it can be shown that to resolve the flow completely, it is only necessary to take into account all modes with eigenvalue λ with a modulus below a maximum $|\lambda_{\max}|$, such that their total number scales as Re^2/Ha [4]. Since the operator $\mathcal{L}$ represents the sum of viscous and Joule dissipation, the set of modes defined in this way is in fact the set of *least dissipative modes*. For sufficiently large values of Ha, this number becomes significantly smaller than the number of Fourier or Tchebychev modes necessary to resolve the Hartmann layers [2].

The main difficulty of solving equation (5) using the least dissipative modes lies in calculating non linear terms. We use a pseudospectral approach and calculate them in real space. Therefore we need a method to reconstruct a spectral coefficients g_n of physical vector fields known at the discrete set of points in space $\mathbf{x}_i$. This problem can be formulated as a set of linear equations for unknown spectral components:

$$\sum_n g_n \mathbf{e}_n(\mathbf{x}_i) = \mathbf{G}(\mathbf{x}_i) \quad i = 1\ldots N \tag{6}$$

where $\mathbf{e}_n$ constitutes are base of eigenvectors, and $\mathbf{G}$ represents the decomposed vector field. As the coefficients in this set of equations are constant during a single numerical run, it is worth performing LU decomposition of the corresponding matrix at the beginning of calculations and

later use it to efficiently find the spectral decompositions. Moreover it enables us to save even more CPU time by omitting calculation of coefficients which wouldn't be used in further calculations. For example we are interested only in the g_n coefficients corresponding to divergence free modes. Neglecting coefficients corresponding to irrotational modes is an equivalent of performing projection representing by operator P in eq. (5). The technique described above has the advantage that the obtained spectral decomposition reproduces exactly the physical field on the given set of discretization points. Therefore momentum and energy are conserved by this procedure.

The spectral method described above was implemented by modifying the spectral code TARANG developed by [7].

To validate the present numerical scheme we compared the results it produces with those of three-dimensional, time-dependent direct numerical simulations performed with a code based on the open source framework OpenFOAM, on test cases of freely decaying MHD turbulence. OpenFOAMis based on the finite volume approach and uses a co-located grid. Our numerical domain was a cube of dimension L divided uniformly into N cells in every direction. In order to calculate correctly the electric current density in Hartman layers we always resolve each of them with at least three computational cells in the z direction as in [1]. Following the DNS of decaying MHD turbulence in a three-dimensional periodic domain by [3], the initial conditions consist of a random gaussian velocity field with $u(k) \sim \exp\left[(-k/k_p)^2\right]$ where $k_p = 8\pi/L$. This corresponds to the energy spectrum $E \sim k^4 \exp\left[-2(k/k_p)^2\right]$. For this choice of initial velocity field, the integral scales of turbulent motions is given by $l = \sqrt{2\pi}/k_p$. The velocity spectrum was normalised in such a way that cell sizes correspond to $l_K/1.4$ where $l_K = lRe^{-3/4}$ is the Kolmogorov length scale and the Reynolds number in its definition $Re = u'l/\nu$ is based on l and velocity $u' = u(k = k_p)$. With this choice, the Reynolds number and the Hartmann number are linked by $Re = 0.33Ha^{4/3}$. This strategy allows us to calculate the most intense flow possible whilst minimizing mesh-induced numerical errors at a given mesh size, since the mesh is always uniform.

For the reference case, we have chosen $Ha = 56$, a value within reach with a traditional code such as OpenFOAM. Adopting the procedure presented in previous paragraph for OpenFOAM calculations we use $N_x = N_y = N_z = 170$ number of points in every direction and initial conditions characterized by Reynolds number $Re = 28$. For the corresponding spectral calculations we use a resolution 1.5 times higher ($N_x = N_y = N_z = 256$) in order to reduce the dealiasing errors. The initial conditions were chosen in such a way that their physical expansion on grid $N_x = N_y = N_z = 170$ was identical to the initial conditions used in the calculations with OpenFOAM.

We have followed the evolution of the initial conditions up to the time corresponding to $30\,t_J$ where $t_J = \sigma B^2/\rho$ is the timescale of Joule dissipation. We have compared the evolution of global kinetic energy, the viscous and magnetic dissipation rates. All these quantities exhibit quantitatively and qualitatively the same behaviour in both codes. The spectral codes exhibits a slightly smaller values of viscous dissipation then the finite volume code and slightly larger magnetic dissipation (see fig. 3).

4 Decaying turbulence

We used the potential of the new method to study the behaviour of the MHD turbulence in two sets of calculations with higher values of the magnetic field: $Ha = 112$ and $Ha = 224$. We started both simulations from exactly the same initial conditions characterized by Reynolds number $Re = 178$ and evolved them up to $60\,t_J$. The fig. 4 shows the evolution of global kinetic energy, the viscous and magnetic dissipation rates. For both values of the magnetic field

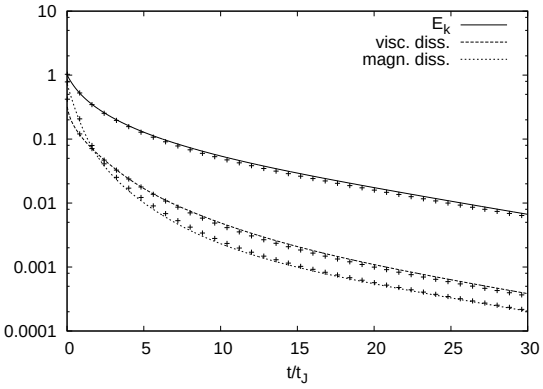

Figure 1: Total kinetic energy (normalized by its initial value) and viscous and magnetic dissipation rates (normalized by initial kinetic energy divided by t_J) in test case with $Ha = 56$ in the function of Joule times. The lines and points respectively represents the results obtained with finite volume and spectral codes.

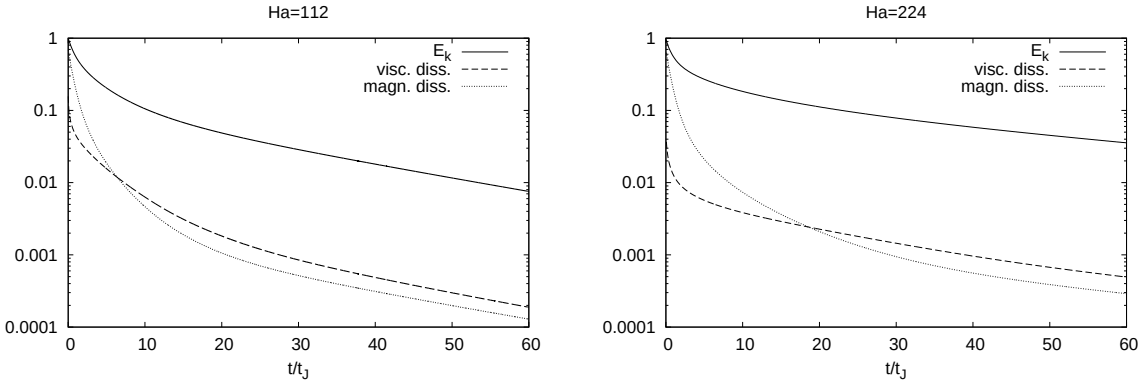

Figure 2: Total kinetic energy (normalized by its initial value) and viscous and magnetic dissipation rates (normalized by initial kinetic energy divided by t_J) in case with insulating walls for $Ha = 112$ (left panel) and $Ha = 224$ (right panel) in the function of Joule times.

the flow evolution can be split into two phases. In the first phase the energy decay is dominated by Joule dissipation. During this phase the flows changes to the state of two-dimensionality due to the diffusion of the momentum along the magnetic field. In the second phase the energy is dissipated mainly by viscosity with the ratio of viscous to magnetic dissipation reaching its maximum value of ~ 1.7 and then slightly decreasing in time. Our results indicates that in this viscously dominated phase the ratio of viscous to Joule dissipations scales as time multiplied by Ha. It is the phase when the flow is strongly two dimensional and its evolution is governed by interaction between 2D vortices and the walls.

To characterize the influence of the walls we performed two additional sets of calculations starting from exactly the same initial conditions as before but with the periodic boundary conditions imposed also in all three directions. The results are presented on fig. 4. In the beginning the dissipation rate is again dominated by Joule dissipation. This phase is very similar to the one in cases with insulating walls. In the second phase the flow is again strongly two dimensional and the energy is dissipated mainly by viscosity. However the Joule dissipation decreases much faster with time with the ratio of viscous to Joule dissipations monotonically increasing with time. Consequently at the end of the simulation the Joule dissipation is negligible, which is in strong contrast to simulations with insulating walls.

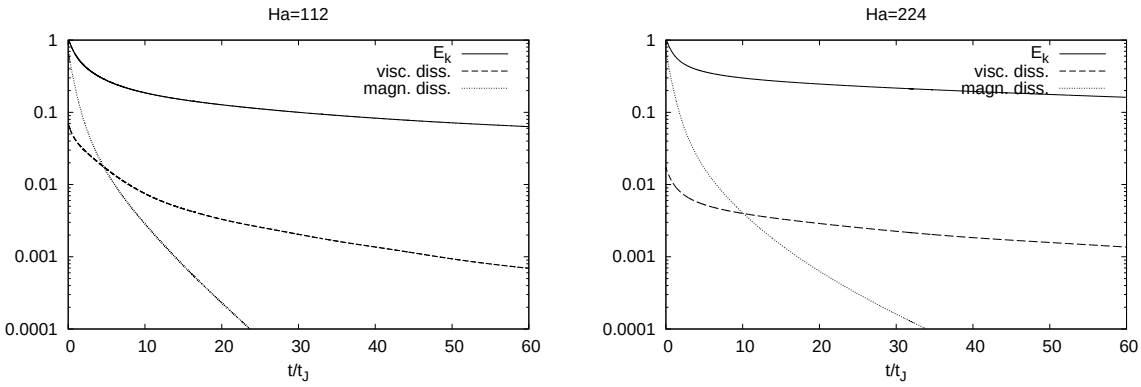

Figure 3: Total kinetic energy (normalized by its initial value) and viscous and magnetic dissipation rates (normalized by initial kinetic energy divided by t_J) in case with fully periodic domain for $Ha = 112$ (left panel) and $Ha = 224$ (right panel) in the function of Joule times.

5 Conclusions

We presented a new spectral method to calculate MHD flows in channel configuration. It is based on using the sequence of least dissipative eigenmodes from the dissipation operator instead of the traditional Fourier or Tchebychev basis. We used this method to calculate the evolution of freely decaying MHD turbulence between walls for $Ha = 112$ and $Ha = 224$. We compared the result with the cases calculated in a 3D periodic domain, which allowed us to quantify the influence of the channel walls on the temporal evolution of viscous and Joule dissipations.

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