INDUCTIONLESS MAGNETOROTATIONAL INSTABILITY BEYOND THE LIU LIMIT

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Abstract: Employing the short wavelength approximation, we develop a unified framework for the investigation of the standard, the helical, and the azimuthal version of the magnetorotational instability (MRI) as well as of the current-driven Tayler instability. We show that the inductionless types of MRI that were previously thought to be restricted to comparably steep rotation profiles extend well to the Keplerian case if only the azimuthal field deviates slightly from its field-free profile.

1. Introduction

How stars and black holes are able to form from rotating matter is one of the big questions of astrophysics. Magnetic fields figure prominently into the picture via the mechanism of magnetorotational instability (MRI) [1,2]. The usual understanding is that MRI only works if matter is electrically well conductive. However, in rotating disks this is not always the case. In areas of low conductivity, like the dead zones of protoplanetary disks or the far-off regions of accretion disks that surround supermassive black holes, the MRI effect is numerically difficult to comprehend and its relevance is thus a matter of dispute. A complementary approach to this regime would be to carry out liquid metal experiments. Unfortunately, under the condition of a purely vertical field, both the rotational speed as well as the magnetic field has to be very high, so that experiments on this standard version of MRI (SMRI) are extremely involved [3,4], and a clear success has eluded them thus far.

By adding an azimuthal magnetic field to the vertical one, as proposed in [5], it became possible to observe the helical MRI (HMRI) at substantially lower rotational speeds and magnetic fields [6]. Very recently, the non-axisymmtric azimuthal MRI (AMRI) has also been observed [7]. However, one of the blemishes of these inductionless versions of MRI is the fact that they are only able to destabilize rotational profiles with a relatively steep radial decay, which for now did not include rotation profiles as shallow as the Keplerian one.

Here, we study the stability of rotational flows in the presence of a constant vertical magnetic field and an azimuthal magnetic field with an arbitrary radial dependence. Employing the short-wavelength approximation, we develop a unified framework for the investigation of SMRI, HMRI, AMRI, as well as of current-driven Tayler instability (TI) [8]. Considering the viscous and resistive case, our main focus is on the limit of small magnetic Prandtl numbers which applies, e.g., to liquid metal experiments but also to the colder parts of accretion disks. We rigorously demonstrate that the inductionless versions of MRI extend well to the Keplerian case if the azimuthal field only slightly deviates from its field-free profile [9-12].

2. Mathematical setting

The standard set of equations of viscous, resistive, incompressible magnetohydrodynamics consists of the Navier-Stokes equation and the induction equation for the time evolution of the fluid velocity **u** and the magnetic field **B,** respectively,

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{\nabla P}{\rho} + \frac{\mathbf{B}\cdot\nabla\mathbf{B}}{\mu_0 \rho} + \nu\nabla^2\mathbf{u} \quad , \qquad \frac{\partial \mathbf{B}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{B} = \mathbf{B}\cdot\nabla\mathbf{u} + \eta\nabla^2\mathbf{B} ,$$

where both **u** and **B** are divergence-free, and P is the total pressure. In the following, we assume a rotating flow with angular velocity $\Omega(r)$, exposed to a magnetic field with constant axial component B_z^0 and an azimuthal magnetic field with arbitrary radial dependence $B_\phi^0(r)$. Around this ground state, we consider short-wavelength perturbations of velocity and magnetic field with an, in general complex, growth rate λ and a wave vector $\mathbf{k}$ for which we use the definitions $\alpha = k_z|\mathbf{k}|^{-1}, |\mathbf{k}|^2 = k_r^2 + k_z^2$. We define the viscous, the resistive, and the two Alfvén frequencies corresponding to the vertical and the azimuthal magnetic field,

$$\omega_\nu = \nu|\mathbf{k}|^2, \quad \omega_\eta = \eta|\mathbf{k}|^2, \quad \omega_{A_z} = \frac{k_z B_z^0}{\sqrt{\rho\mu_0}}, \quad \omega_{A_\phi} = \frac{B_\phi^0}{r\sqrt{\rho\mu_0}},$$

and measure the radial steepness of the angular velocity and the azimuthal Alfvén frequency by two appropriate Rossby numbers:

$$\mathrm{Ro} = \frac{r}{2\Omega}\frac{\partial\Omega}{\partial r}, \quad \mathrm{Rb} = \frac{r}{2\omega_{A_\phi}}\frac{\partial\omega_{A_\phi}}{\partial r}.$$

After introducing the following dimensionless numbers,

$$\mathrm{Pm} = \frac{\omega_\nu}{\omega_\eta}, \quad \beta = \alpha\frac{\omega_{A_\phi}}{\omega_{A_z}}, \quad \mathrm{Re} = \alpha\frac{\Omega}{\omega_\nu}, \quad \mathrm{Ha} = \frac{\omega_{A_z}}{\sqrt{\omega_\nu\omega_\eta}}, \quad n = \frac{m}{\alpha},$$

we can derive the secular equation for the perturbations in the form

$$p(\lambda) = \det(\mathbf{H} - \lambda\mathbf{E}) = 0,$$

with the matrix

$$\mathbf{H} = \begin{pmatrix} -in\mathrm{Re}-1 & 2\alpha\,\mathrm{Re} & \frac{i\mathrm{Ha}(1+n\beta)}{\sqrt{\mathrm{Pm}}} & -\frac{2\alpha\beta\mathrm{Ha}}{\sqrt{Pm}} \\ -\frac{2\,\mathrm{Re}(1+\mathrm{Ro})}{\alpha} & -in\mathrm{Re}-1 & \frac{2\beta\mathrm{Ha}(1+\mathrm{Rb})}{\alpha\sqrt{\mathrm{Pm}}} & \frac{i\mathrm{Ha}(1+n\beta)}{\sqrt{\mathrm{Pm}}} \\ \frac{i\mathrm{Ha}(1+n\beta)}{\sqrt{\mathrm{Pm}}} & 0 & -in\mathrm{Re}-\frac{1}{\mathrm{Pm}} & 0 \\ \frac{-2\beta\mathrm{Ha}\mathrm{Rb}}{\alpha\sqrt{\mathrm{Pm}}} & \frac{i\mathrm{Ha}(1+n\beta)}{\sqrt{\mathrm{Pm}}} & \frac{2\,\mathrm{Re}\,\mathrm{Ro}}{\alpha} & -in\mathrm{Re}-\frac{1}{\mathrm{Pm}} \end{pmatrix}.$$

This gives a dispersion relation in form of complex fourth-order polynomial, for which Bilharz's stability criterion can be applied [10-12].

3. Some results

Soon after the discovery of the HMRI [5], and its surprising scaling with the Reynolds and Hartmann numbers (which is different from SMRI that scales with the magnetic Reynolds and Lundquist number), two limits were identified by Liu et al. [13] which we will call the Lower Liu Limit (LLL) and the Upper Liu Limit (ULL) in the following. In the inductionless limit Pm = 0, and for a current-free azimuthal field, i.e. Rb = −1, the authors had found that the flow is stable for Rossby numbers between the LLL $\mathrm{Ro}_{\mathrm{LLL}} = 2(1-2^{1/2}) = -0.8284$ and the ULL $\mathrm{Ro}_{\mathrm{ULL}} = 2(1+2^{1/2}) = +4.8284$. The existence of the LLL, in particular, is of great astrophysical relevance since it means that Keplerian rotations, characterized by Ro = −3/4, would not be affected by HMRI (and neither by AMRI, as was later shown in [9]). Despite some attempts

to extend the LLL to somewhat higher values (by considering conducting boundaries [14,15] or finite Pm [16]), it seems now that Keplerian profiles will be very hard to be destabilized.
Here, we discuss another way of extending the range of applicability of HMRI (and AMRI). We set out from the physical reasoning that the shape of the azimuthal magnetic field in a disk is not a-priori given, but is rather a product of induction effects in the disk. The ~1/r dependence would correspond to the extreme case of an axial current in the very center of the disk. Without going into the details of induction effects, which would depend strongly on the radial and vertical distributions of the conductivity, we assume here that the azimuthal field might well be flatter than 1/r, and we will test the consequences of this modification for the applicability of HMRI.

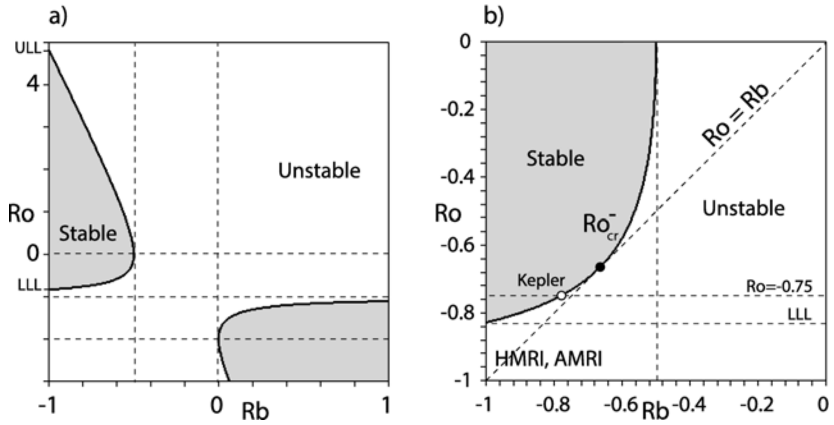

Figure 1: (a) The lower (LLL) and the upper (ULL) Liu limits existing at Rb = −1 are just the end points of a quasi-hyperbolic curve in the Ro−Rb plane. (b) A scaled fragment of the limiting curve demonstrating that the inductionless forms of the MRI can exist above the limit $Ro_{LLL} = 2(1\text{-}2^{1/2})$ in case that Rb > −1. The open circle marks the Keplerian value with Ro = −3/4 at Rb = −25/32, whereas the black circle corresponds to Ro = Rb = −2/3. The dashed diagonal represents the Chandrasekhar line Ro = Rb.

Analyzing the secular equation for arbitrary Ro and Rb, assuming Pm = 0, letting Re and Ha go to infinity, and optimizing than over β, we obtain the following curve of marginal stability in the Ro-Rb plane [10]:

$$\mathrm{Rb} = -\frac{1}{8}\frac{(\mathrm{Ro}+2)^2}{\mathrm{Ro}+1}\,.$$

This curve is visualized in Figure1. In Figure 1a we see that the two Liu limits LLL and ULL are just the endpoints of this curve. Most important for us is the fact that the Keplerian case, i.e. Ro = −0.75, is reached at Rb = −25/32 = −0.78125. Roughly speaking, a slight 20 per cent deviation from the purely current-free azimuthal field (Rb = −1), would make HMRI a viable mechanism to destabilize Keplerian flows.
A second interesting point in Figure 1b is Ro = Rb = −2/3. This is the only point where our marginal stability curve touches the so-called Chandrasekhar line, characterized by Ro = Rb which means that the angular velocity and the azimuthal Alfvén frequency have the same radial dependence. The particular Chandrasekhar equipartition solution, with Ro = Rb = −1, is known to be stable in the ideal case. What happens with the general line Ro = Rb, when viscosity and resistivity come into play? To answer this question we set the interaction parameter (or Elsasser number) N = Ha^2/Re equal to the magnetic Reynolds number, N = Rm. Under this condition, the Bilharz criterion acquires the form

$$16(n^2-\mathrm{Rb}^2)(n^2-\mathrm{Rb}-2)^2\mathrm{Rm}^4+(n^6-12n^2\mathrm{Rb}^2+32n^2(\mathrm{Rb}+1)-16\mathrm{Rb}^2(\mathrm{Rb}+2))\mathrm{Rm}^2$$
$$-4Rb^2+4n^2(Rb+1)=0$$

Its solution is illustrated in Figure 2. The wide part of the instability domain in Figure 2a, existing for small Rm, represents the AMRI. Evidently, this domain shrinks with increasing Rm, degenerating to a ray for infinite Rm. In other words, when coming from infinite Rm, already an infinitesimal small electrical resistivity destabilizes the marginal stable solution. In this sense, AMRI is a typical example of a dissipation-induced instability.

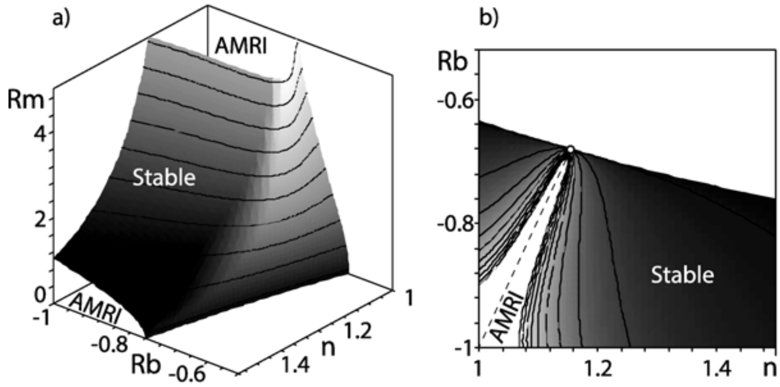

Figure 2: (a) The threshold of instability at $\mathrm{Ha}^2/\mathrm{Re} = \mathrm{Rm}$ and $\mathrm{Re} \to \infty$ in the (n, Rb, Rm) space. (b) Its projection onto the Rb−n plane. The increase in Rm makes the instability domain more narrow so that in the limit $\mathrm{Rm} \to \infty$ it degenerates into a ray (dashed) that emerges from the point (open circle) with the coordinates $n = 2/3^{1/2}$ and $\mathrm{Rb} = -2/3$ and passes through the point with n = 1 and Rb = −1.

Going over from Rb = −1 to Rb = 0 means physically a transition from an isolated central current to a homogeneous radial current distribution. The latter situation is known to be susceptible to the Tayler instability [8], which tapes into the energy of the current rather than into the rotational energy. Setting Hb: = β Ha, with Ha going to 0 and β to infinity, we obtain

$$\mathrm{Re}^2=\frac{((1+\mathrm{Hb}^2n^2)^2-4\mathrm{Hb}^2\mathrm{Rb}(1+\mathrm{Hb}^2n^2)-4\mathrm{Hb}^4n^2)(1+\mathrm{Hb}^2(n^2-2\mathrm{Rb}))^2}{4(\mathrm{Hb}^4\mathrm{Ro}^2n^2-((1+\mathrm{Hb}^2(n^2-2\mathrm{Rb}))^2-4\mathrm{Hb}^4n^2)(\mathrm{Ro}+1))}\ .$$

Figure 3 shows now the stability surface for this setting, at Pm = 0 and n = 1.278, whereby we connect Rb = −1 and Rb = 0 by a quarter of a circle according to $\mathrm{Ro(Rb)} = -(-\mathrm{Rb}^2 - 2\mathrm{Rb})^{1/2}$. Figure 3a gives a total view of this surface, while Figures 3b-d show individual slices at different values of Re. Not surprisingly, at Re = 0 we get only the current-driven TI, while for Re > 0 we see the AMRI arising as a “nose” which later connects to the TI area.

4. Conclusions

Given the dramatic differences in the parameter dependencies of SMRI on one side and HMRI/AMRI on the other side, it is of great astrophysical importance to know whether the latter forms could possibly be working for Keplerian rotation profiles. As we have seen, the answer to this question is affirmative, if the azimuthal magnetic field is only slightly shallower than ~1/r. Yet, the induction that is needed to allow this would require Rm > 1 which apparently leads us back to the realm of SMRI. However, there is still a difference here since the Lundquist number could be very small in our case. Detailed considerations for specific accretion disk problems must be left for future work.

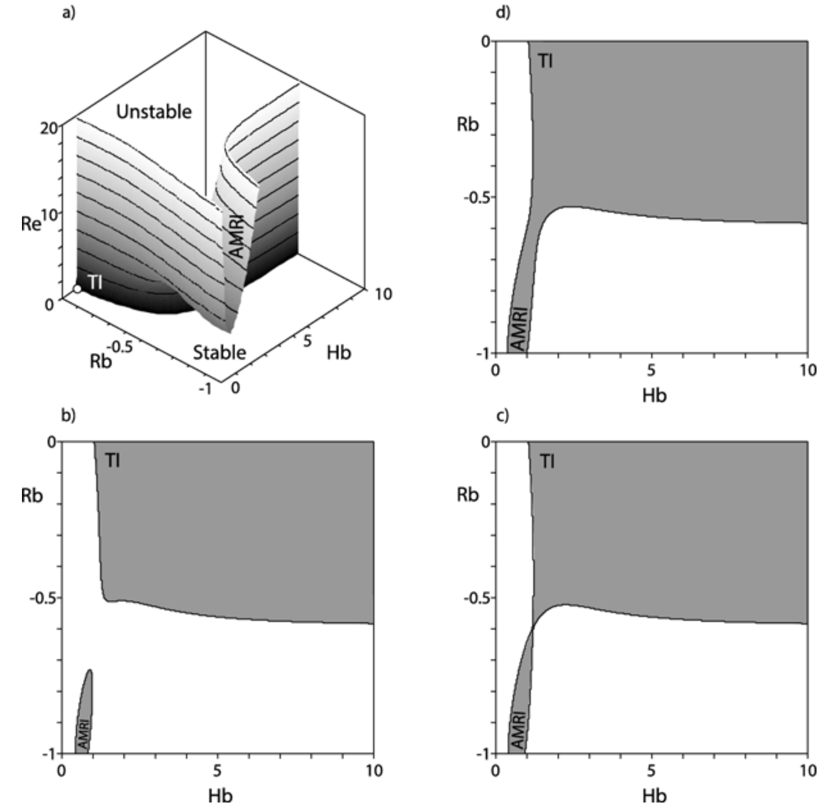

Figure 3: Instability threshold for the special case Pm = 0, and n ≈ 1.27842, when following the quarter-circle curve $Ro(Rb) = -(-Rb^2 - 2Rb)^{1/2}$. (a) The instability domain bounded in the (Hb,Rb,Re) space and its cross-sections at (b) Re = 5.4, (c) Re = 5.734, and (d) Re = 6. The domains of TI and AMRI reconnect via a saddle point at Re = 5.734.

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