Stability boundaries of axisymmetric and two-dimensional perturbations in MHD Dean flow

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Abstract. We study the linear stability of annular MHD channel flow with a uniform axial magnetic field in order to determine when two-dimensional instabilities of Orr type can appear. They are a prerequisite for intermittent turbulent behavior known from plane MHD channel flow with a spanwise field. The annular flow is driven by Lorentz forces caused by a radial electric current and the imposed axial field. Stability of this MHD Dean flow is investigated for axially uniform Orr modes and axisymmetric Dean modes. Orr mode instability dominates only for small gap width and in strong magnetic fields.

Introduction. The interaction of flows of liquid metals and other conducting liquids with magnetic fields can be used for flow control or measurement purposes in metallurgy and other materials processing applications. Typically, the magnetic field has a damping effect in such MHD flows. The flow structures in wall-bounded MHD flows are also modified due to electromagnetic boundary layers. Transition to turbulence and the properties of turbulent MHD flows can therefore differ significantly from non-MHD flows. Such questions have been explored by computational studies in recent years since experiments are difficult and typically provide only very limited information on such flows.

The selective damping of gradients in the flow along the direction of the magnetic field can also favor flow instabilities that are otherwise superseded by other processes. An example is the so-called large-scale intermittency (LSI) in plane MHD channel flow with a homogeneous spanwise field found in a number of computational studies [1, 2]. In this problem, the viscous instability of Orr modes (also called Tollmien-Schlichting modes) is unaffected by the magnetic damping but the growth of non-modal perturbations normally causing bypass transition is suppressed. One therefore finds a cyclic evolution between the unstable laminar state and turbulent flow that is quickly suppressed by Joule dissipation. Experimental verification of this phenomenon is missing so far and may be difficult to achieve due to the additional friction at Hartmann walls that are not taken into account in channel flow simulations of LSI. Another problem is the generation of a sufficiently strong and homogeneous field over a sufficient length of a straight channel. Experimentally it is preferable to use an annular channel that can be placed in the bore of a solenoid magnet, which generates an axial field. The flow can then be driven by an azimuthal Lorentz force that results from applying a radial current between the cylindrical walls of the annular channel [3]. The problem in this setup is the presence of centrifugal instabilities that may supersede the Orr mode instability. In the present paper we therefore consider the magnetic damping of such centrifugal instabilities. The goal is to identify parameters where Orr mode instability prevails so that the LSI could be observed at least in principle.

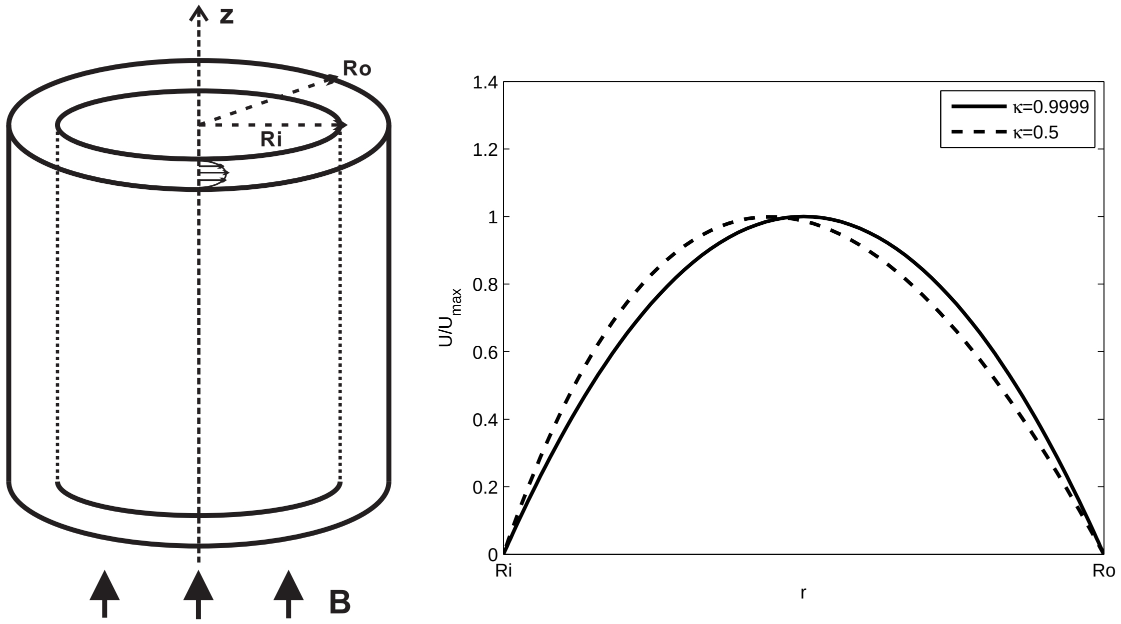

Figure 1: Sketch of the geometry (left) and basic flow structure (right)

Previous studies have focused on the narrow gap approximation [4]. We consider the problem without making this assumption. End walls have to be neglected in order to obtain one-dimensional stability problems that can be solved with modest computational expense.

Governing equations and parameters. We consider the flow of an incompressible electrically conducting fluid with conductivity σ, density ρ and kinematic viscosity ν in the gap between two concentric cylinders. A potential difference is imposed between the cylinders, which are assumed to be perfectly conducting. The driving Lorentz force results from the interaction between the imposed uniform axial magnetic field $\boldsymbol{B}_0 = B_0\boldsymbol{e}_z$ and the radial electric current. The base flow has only an azimuthal velocity component, which depends on the radial coordinate. End walls in the axial direction are not taken into account. Fig. 1 shows the basic geometry schematically.

The governing equations for velocity $\boldsymbol{u}$ and electric potential ϕ in the quasistatic approximation are the Navier-Stokes equations together with Ohm's law for the current density $\boldsymbol{j}$ and charge conservation, i.e.

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\boldsymbol{u} + \frac{1}{\rho}\boldsymbol{j}\times\boldsymbol{B}_0, \tag{1}$$

$$\nabla\cdot\boldsymbol{u} = 0, \tag{2}$$

$$\boldsymbol{j} = \sigma(-\nabla\phi + \boldsymbol{u}\times\boldsymbol{B}_0), \tag{3}$$

$$\nabla\cdot\boldsymbol{j} = 0 \leftrightarrow \nabla^2\phi = \nabla\cdot(\boldsymbol{u}\times\boldsymbol{B}_0)\,. \tag{4}$$

In the following we use cylindrical coordinates r, θ and z and corresponding vector components. Boundary conditions are zero velocity and fixed electric potentials ϕ_i and ϕ_o at the inner and outer walls located at $r = R_i$ and $r = R_o$. For the presentation of results we introduce the ratio

$$\kappa = R_i/R_o \tag{5}$$

to characterize the influence of curvature with $\kappa = 1$ corresponding to plane channel flow.

The Reynolds number

$$Re = \bar{U}d/\nu \tag{6}$$

is defined with the average azimuthal velocity $\bar{U}$ and the gap width $d = R_o - R_i$ Finally, the Hartmann number is defined by

$$Ha = dB_0\sqrt{\sigma/\rho\nu}. \tag{7}$$

The basic velocity distribution of this so-called Dean flow has only the azimuthal component $u_\theta = V$ that depends on r and κ. It is given by

$$V(\eta) = \bar{U}C(\kappa)\left\{\eta\log\eta + \frac{\kappa^2\log\kappa}{1-\kappa^2}\left[\eta - \eta^{-1}\right]\right\}, \tag{8}$$

where $\eta = r/R_o$ and C is determined by the imposed volume flux. Fig. 1 shows that $V(\eta)$ is asymmetric and that it approaches the parabolic profile for $\kappa \to 1$.

The linear stability analysis using normal modes requires linearization of eq. (1-4) about $V(\eta)$. Normal modes are periodic in both θ and z, i.e. they are exponentials

$$\exp(i\omega t)\exp(im\theta)\exp(i\alpha z) \tag{9}$$

in θ, z and time t with complex frequency ω, an integer azimuthal wave number m and a real axial wavenumber α.

We are not interested in general perturbations with both m and α non-zero. For the non-magnetic case the flow typically becomes first unstable to axisymmetric perturbations ($m = 0$) called Dean modes. Intermittency in plane MHD channel flow requires a linear instability to Orr modes unaffected by the magnetic field, i.e. with $\alpha = 0$. We want to find conditions where similar behavior may be obtained in direct numerical simulations (DNS) of the Dean flow, i.e. parameter combinations of κ, Re, and Ha where the basic flow is only unstable with respect to the Orr mode. We therefore determine the neutral stability limits of Orr modes with $\alpha = 0$ and Dean modes with $m = 0$.

Two-dimensional Orr modes ($\alpha = 0$) have radial and azimuthal components that can be represented by a stream function $\psi = f(r)\exp(im\theta + i\omega t)$. Dean modes have all three velocity components. Each of them takes the form $g(r)\exp(i\alpha z + i\omega t)$. In either case, we use an expansion in Chebyshev polynomials for discretization in the radial coordinate. A linear algebraic eigenvalue problem for ω is then obtained by means of a Chebyshev collocation method and solved in MATLAB using the QZ algorithm. Neutral conditions are then determined by varying either Re or Ha until $\omega_i = 0$ is achieved.

An alternative approach is possible by DNS, which have also been used for stability studies in the same geometry [5].

Results and discussion. We first consider the Orr modes, which depend on κ and Re. In plane channel (Poiseuille) flow Orr modes become unstable at $Re_P^c \approx 5772$ with a nondimensional wavenumber $\alpha_P^c = 1.02$ based on $d/2$, i.e. a dimensional wavelength of $d\pi/\alpha_P^c$ [6]. These values should be recovered in the limit $\kappa \to 1$. We have therefore started our computations near $\kappa = 1$. In contrast to plane channel flow the wavenumber m has to be integer. This leads to significant changes when κ becomes small since the values of m are then fairly low. The results are shown in Fig. 2. We see that Re_c increases as κ is reduced until it reaches a maximum near $\kappa = 0.55$. The subsequent decrease is due to a switch from $m = 3$ to $m = 2$. Below $\kappa \approx 0.45$ the $m = 2$ branch does not exist. Fig. 2 also shows Re_c and the wavenumber m for plane Poiseuille flow for comparison with the Dean

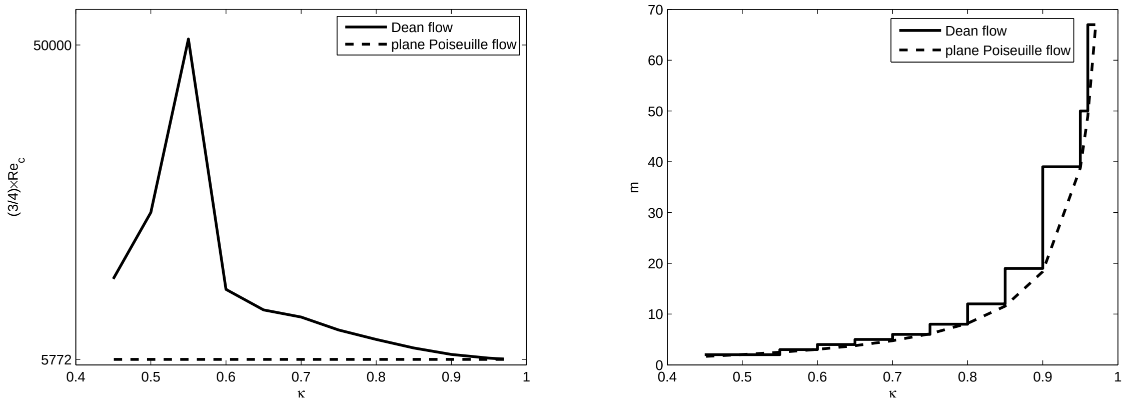

Figure 2: Stability limits of Orr modes (left) and corresponding azimuthal wavenumbers m (right) as function of the radius ratio κ. In plane channel flow the Reynolds number is based on the maximum velocity and the channel half-width, hence the prefactor 3/4.

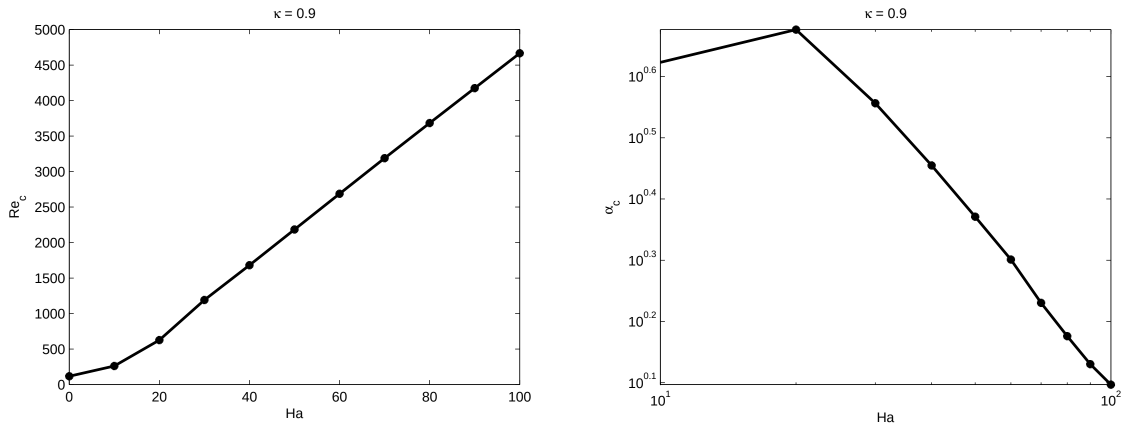

Figure 3: Critical Reynolds number of Dean mode for $\kappa = 0.9$ (left) and corresponding axial wavenumber (right).

flow. The wavenumber m for Poiseuille flow has been computed as $m = 2\alpha_c^P\kappa/(1-\kappa)$, i.e. as the ratio of the the circumference of the inner cylinder to the critical wavelength $d\pi/\alpha_P^c$. We see that this provides a fairly good estimation of m except for low $\kappa < 0.6$. The discontinuous changes in m for the Dean flow should normally proceed in increments $\Delta m = 1$. The larger jumps in Fig. 2 are due to the finite number of κ values explored in our computations.

For Dean modes the non-magnetic stability limit Re_c has been computed in the narrow-gap approximation. It is given by [4]

$$Re_c \approx 35.94\sqrt{\kappa/(1-\kappa)}, \qquad \alpha_c \approx 3.96, \tag{10}$$

where the nondimensional wavenumber α_c is based on the length d. We have first verified these results for κ close to unity. Finite values of Ha lead to a monotonous increase in Re_c due to the magnetic damping. It becomes linear in Ha when Ha is sufficiently large, i.e. the slope dRe_c/dHa is constant. The corresponding axial wave numbers α_c decrease as $1/Ha$. Both effects are illustrated in Fig. 3. In contrast to Re_c, α_c hardly change with κ. However, at fixed Ha Re_c increases strongly with κ.

Based on our results, we have determined regions in the *Re*-*Ha* plane where Dean

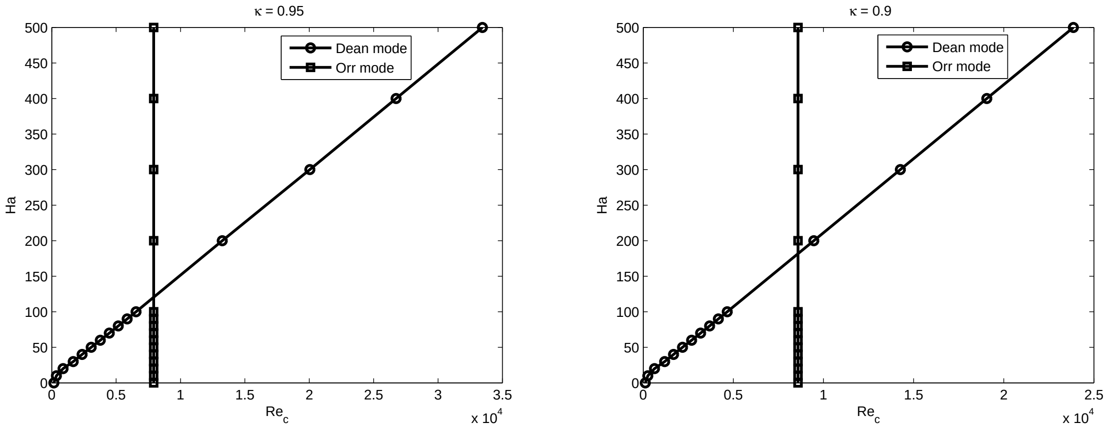

Figure 4: Stability limits in the Re-Ha-plane for Orr modes and Dean modes for $\kappa = 0.9$ (left) and $\kappa = 0.95$ (right).

modes are sufficiently suppressed by the magnetic damping so that the Orr modes become unstable first. Fig. 4 shows two different values of κ. Orr modes are unstable for $Re > Re_c^O$ irrespective of Ha. The Dean modes are unstable for $Re > Re_c^D(Ha)$, i.e. below the corresponding limiting curve in the Re-Ha-plane. The region of Orr mode instability lies therefore in the upper triangular section of the plane. The minimum values of Ha are substantial and increase strongly when κ is reduced.

Conclusions. The Dean mode instability can only be suppressed in favor of Orr modes when the gap width is small and the Hartmann number is substantial. For this reason, experiments in a parameter range where LSI could potentially exist would require a substantial diameter of the annular channel as well as fairly strong fields. Investigation of the effects due to end walls requires DNS at large values of κ and high aspect ratios of axial size to gap width.

Acknowledgments. We are grateful to O. Zikanov and D. Krasnov for interesting discussions. YY's work is supported by the China Scholarship Council. TB acknowledges financial support from the Deutsche Forschungsgemeinschaft.

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