Two-dimensional nonlinear travelling waves in MHD channel flow

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Abstract: The present study is concerned with the stability of a flow of viscous conducting liquid driven by pressure gradient in the channel between two parallel walls subject to a transverse magnetic field. Although the magnetic field has a strong stabilizing effect, this flow, similarly to its hydrodynamic counterpart – plane Poiseuille flow, is known to become turbulent significantly below the threshold predicted by linear stability theory. We investigate the effect of the magnetic field on 2D nonlinear travelling-wave states which are found at substantially subcritical Reynolds numbers starting from $Re_n = 2939$ without the magnetic field and from $Re_n \sim 6.50 \times 10^3 Ha$ in a sufficiently strong magnetic field defined by the Hartmann number Ha. Although the latter value is by a factor of seven lower than the linear stability threshold $Re_l \sim 4.83 \times 10^4 Ha$, it is still more by an order of magnitude higher than the experimentally observed value for the onset of turbulence in this flow.

1 Introduction

The flow of viscous incompressible liquid driven by a constant pressure gradient in the channel between two parallel walls, which is generally known as plane Poiseuille or simply channel flow, is one of the simplest and most extensively studied models of hydrodynamic instabilities and transition to turbulence in shear flows. The development of turbulence in the magnetohydrodynamic (MHD) counterpart of this flow, which is known as Hartmann flow and arises when a conducting liquid flows in the presence of a transverse magnetic field, is currently not so well understood. The MHD channel flow, which was first described theoretically by Hartmann [1] is still an active subject of research [2, 3].

The present study is concerned with finding such 2D travelling-wave states in Hartmann flow. Starting from plane Poiseuille flow, we trace such subcritical equilibrium states by gradually increasing the magnetic field. Using an accurate numerical method based on Chebyshev collocation approximation and a sufficiently large number of harmonics we find such states extend to subcritical Reynolds number $R_n \approx 6500$ which is almost by a factor of two smaller than that predicted by the mean-field approximation [5].

2 Formulation of problem

Consider the flow of an incompressible viscous electrically conducting liquid with density ρ, kinematic viscosity ν and electrical conductivity σ driven by a constant gradient of pressure p in the channel of the width $2h$ between two parallel walls in the presence of a transverse homogeneous magnetic field $\vec{B}$. The velocity distribution of the flow $\vec{v}(\vec{r},t)$ is governed by the Navier-Stokes equation $\partial_t\vec{v} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\rho^{-1}\vec{\nabla}p + \nu\vec{\nabla}^2\vec{v} + \rho^{-1}\vec{f}$, where $\vec{f} = \vec{j}\times\vec{B}$ is the electromagnetic body force containing the induced electric current $\vec{j}$, which is governed by Ohm's law for a moving medium $\vec{j} = \sigma(\vec{E} + \vec{v}\times\vec{B})$, where $\vec{E}$ is the electric field in the stationary frame of reference. The flow is assumed to be sufficiently slow that the induced magnetic field is negligible relative to the imposed one. This supposes a small magnetic Reynolds number

$Re_m = \mu_0 \sigma v_0 h \ll 1$, where μ_0 is the permeability of vacuum and v_0 is the characteristic velocity of the flow. In addition, we assume that the characteristic time of velocity variation is much longer than the magnetic diffusion time $\tau_m = \mu_0 \sigma h^2$. This allows us to use the quasi-stationary approximation leading to $\vec{E} = -\vec{\nabla}\phi$, where ϕ is the electrostatic potential. The velocity and current satisfy mass and charge conservation $\vec{\nabla}\cdot\vec{v} = \vec{\nabla}\cdot\vec{j} = 0$. Applying the latter to the Ohm's law yields $\vec{\nabla}^2\phi = \vec{B}\cdot\vec{\omega}$, where $\vec{\omega} = \vec{\nabla}\times\vec{v}$ is vorticity. At the channel walls S, the normal (n) and tangential (τ) velocity components satisfy the impermeability and no-slip boundary conditions $v_n|_s = 0$ and $v_\tau|_s = 0$. Electrical conductivity of the walls is irrelevant for the type of flow considered in this study.

We employ right-handed Cartesian coordinates with the origin set at the mid-height of the channel, the x- and the z-axes directed, respectively, against the applied pressure gradient $\vec{\nabla}p_0 = P\vec{e}_x$ and along the magnetic field $\vec{B} = B\vec{e}_z$ so that the channel walls are located at $z = \pm h$, and the velocity is defined as $\vec{v} = (u, v, w)$. Subsequently, all variables are non-dimensionalised by using h, h^2/ν and $Bh\nu$ as the length, time and electric potential scales, respectively. The velocity is scaled by the viscous diffusion speed ν/h, which we employ as the characteristic velocity instead of the commonly used centreline velocity.

The problem admits a rectilinear base flow $\vec{v}_0(z) = \bar{u}_0(z)\vec{e}_x = Re\,\bar{u}(z)\vec{e}_x$ for which the Navier-Stokes equation reduces to $\bar{u}'' - Ha^2\bar{u} = \bar{P}$, where $Re = Uh/\nu$ is the Reynolds number based on the centreline velocity U, $Ha = dB\sqrt{\sigma/\rho\nu}$ is the Hartmann number, and $\bar{P}$ is a dimensionless pressure gradient satisfying the normalization condition $\bar{u}(0) = 1$. This equation defines the well-known Hartmann flow profile $\bar{u}(z) = \frac{\cosh(Ha)-\cosh(zHa)}{\cosh(Ha)-1}$ with $\bar{P} = -\frac{Ha^2\cosh(Ha)}{\cosh(Ha)-1}$, which relates the centreline velocity with the applied pressure gradient $P = \bar{P}U\nu\rho/h^2$. In the weak magnetic field $(Ha \ll 1)$, the Hartman flow reduces to the classic plane Poiseuille flow $\bar{u}(z) = 1 - z^2$.

3 Theoretical background

3.1 Linear stability of the base flow

The two-dimensional travelling waves considered this study are expected to emerge in the result of linear instability of the Hartmann flow with respect to infinitesimal perturbations $\vec{v}_1(\vec{x}, t)$. Owing to the invariance of the base flow in both t and $\vec{x} = (x, y)$, perturbations are sought as Fourier modes $\vec{v}_1(\vec{r}, t) = \hat{\vec{v}}(z)e^{\lambda t + i\vec{k}\cdot\vec{x}} + \text{c.c.}$ defined by complex amplitude distribution $\hat{\vec{v}}(z)$, temporal growth rate λ and the wave vector $\vec{k} = (\alpha, \beta)$. The incompressibility constraint, which takes the form $\vec{D}_k \cdot \hat{\vec{v}} = 0$, where $\vec{D}_k \equiv \vec{e}_z \frac{\mathrm{d}}{\mathrm{d}z} + i\vec{k}$ is a spectral counterpart of the nabla operator, is satisfied by expressing the component of the velocity perturbation in the direction of the wave vector as $\hat{u}_{\shortparallel} = \vec{e}_{\shortparallel} \cdot \hat{\vec{v}} = ik^{-1}\hat{w}'$, where $\vec{e}_{\shortparallel} = \vec{k}/k$ and $k = |\vec{k}|$. Taking the *curl* of the linearised Navier-Stokes equation to eliminate the pressure gradient and then projecting it onto $\vec{e}_z \times \vec{e}_{\shortparallel}$, after some transformations we obtain a modified Orr-Sommerfeld type equation which includes a magnetic term

$$\lambda \vec{D}_k^2 \hat{w} = \left[\vec{D}_k^4 - Ha^2(\vec{e}_z \cdot \vec{D}_k)^2 + ikRe(\bar{u}'' - \bar{u}\vec{D}_k^2)\right]\hat{w}. \quad (1)$$

The no-slip and impermeability boundary conditions require $\hat{w} = \hat{w}' = 0$ at $z = \pm 1$. The equation above is written in a non-standard form corresponding to our choice of the characteristic velocity. Note that the Reynolds number appears in this equation as a factor at the convective term rather than its reciprocal at the viscous term as in the standard form. As a result, the growth rate λ differs by a factor Re from its standard definition.

Since the equation above as its non-magnetic counterpart admits Squire's transformation, in the following we consider only two-dimensional perturbations ($k=\alpha$), which are the most unstable. The problem is solved numerically using the Chebyshev collocation method which is described in detail in Ref. [4].

3.2 Nonlinear 2D travelling waves

Two-dimensional travelling waves emerge as follows. First, the neutrally stable mode with a purely real frequency $\omega=-\mathrm{i}\lambda$ interacting with itself through the quadratically nonlinear term in the Navier-Stokes equation produces a steady streamwise invariant perturbation of the mean flow as well as a second harmonic $\sim \mathrm{e}^{2\mathrm{i}(\omega t+\alpha x)}$. Further nonlinear interactions produce higher harmonics, which similarly to the fundamental and second harmonic travel with the same phase speed $c=-\omega/\alpha$. Thus, the solution can be sought in the form$\vec{v}(\vec{r},t)=\sum_{n=-\infty}^{\infty}E^n\vec{v}_n(z)$,where $E=\mathrm{e}^{\mathrm{i}(\omega t+\alpha x)}$ contains ω, which needs to determined together $\vec{v}_n$ by solving a non-linear eigenvalue problem. The reality of solution requires $\vec{v}_{-n}=\vec{v}_n^*$, where the asterisk stands for the complex conjugate. The incompressibility constraint applied to the nth velocity harmonic results in $\vec{D}_{\alpha_n}\cdot\vec{v}_n=0$, where $\vec{D}_{\alpha_n}\equiv\vec{e}_z\frac{\mathrm{d}}{\mathrm{d}z}+\mathrm{i}\vec{e}_x\alpha_n$ with $\alpha_n=\alpha n$ stands for the spectral counterpart of the nabla operator. This constraint can be satisfied by expressing the streamwise velocity component $\hat{u}_n=\vec{e}_x\cdot\vec{v}_n=\mathrm{i}\alpha_n^{-1}\hat{w}_n'$ in terms of the transverse component $\hat{w}_n=\vec{e}_z\cdot\vec{v}_n$, which we employ instead of the commonly used stream function. Henceforth, the prime is used as a shorthand for d/dz. Note that the previous expression is not applicable to the zeroth harmonic, for which it yields $\hat{w}_0\equiv 0$. Thus, $\hat{u}_0$ needs to be considered separately in this velocity-based formulation.

Taking the *curl* of the Navier-Stokes equation to eliminate the pressure gradient and then projecting it onto $\vec{e}_y$, we obtain

$$[\vec{D}_{\alpha_n}^2-\mathrm{i}\omega n]\hat{\zeta}_n-Ha^2\hat{u}_n'=\hat{h}_n, \tag{2}$$

where

$$\hat{\zeta}_n=\vec{e}_y\cdot\vec{D}_{\alpha_n}\times\vec{v}_n=\begin{cases}\mathrm{i}\alpha_n^{-1}\vec{D}_{\alpha_n}^2\hat{w}_n, & n\neq 0;\\ \hat{u}_0', & n=0.\end{cases} \tag{3}$$

and $\hat{h}_n=\sum_m\vec{v}_{n-m}\cdot\vec{D}_{\alpha_m}\hat{\zeta}_m$ are the y-components of the nth harmonic of the vorticity $\vec{\zeta}=\vec{\nabla}\times\vec{v}$ and that of the *curl* of the nonlinear term $\vec{h}=\vec{\nabla}\times(\vec{v}\cdot\vec{\nabla})\vec{v}$. Henceforth, the omitted summation limits are assumed to be infinite. Separating the terms involving $\hat{u}_0$, the previous expression for h can be rewritten as $\hat{h}_n=\mathrm{i}\alpha_n^{-1}(\hat{h}_n^w+\hat{h}_n^u)$, where

$$\hat{h}_n^w = n\sum_{m\neq 0}m^{-1}(\hat{w}_{n-m}\vec{D}_{\alpha_m}^2\hat{w}_m'-\hat{w}_m'\vec{D}_{\alpha_{n-m}}^2\hat{w}_{n-m}), \tag{4}$$

$$\hat{h}_n^u = \mathrm{i}\alpha_n[\hat{u}_0-\hat{u}_0''\vec{D}_{\alpha_n}^2]\hat{w}_n\equiv\mathscr{N}_n(\hat{u}_0)\hat{w}_n. \tag{5}$$

Eventually, using the expressions above, (2) can be written as $\mathscr{L}_n(\mathrm{i}\omega,\hat{u}_0)\hat{w}_n=\hat{h}_n^w$ with the operator

$$\mathscr{L}_n(\mathrm{i}\omega,\hat{u}_0)=[\vec{D}_{\alpha_n}^2-\mathrm{i}\omega n]\vec{D}_{\alpha_n}^2-Ha^2(\vec{e}_z\cdot\vec{D}_{\alpha_n})^2-\mathscr{N}_n(\hat{u}_0). \tag{6}$$

This equation governs all harmonics except the zeroth one, for which, in accordance with the incompressibility constraint it implies $\hat{w}_0\equiv 0$. Zeroth velocity harmonic, which has only the streamwise component $\hat{u}_0$, is governed directly the x-component of the Navier-Stokes equation: $\hat{u}_0''-Ha^2\hat{u}_0=\hat{P}_0+\hat{g}_0$,where $\hat{P}_0=\bar{P}Re$ is a dimensionless mean pressure gradient and $\hat{g}_0=$

$i\sum_{m\neq 0}\alpha_m^{-1}\hat{w}_m^*\hat{w}_m''$ is the x-component of the zeroth harmonic of the nonlinear term $\vec{g} = (\vec{v}\cdot\vec{\nabla})\vec{v}$. Velocity harmonics are subject to the usual no-slip and impermeability boundary conditions $\hat{w}_n = \hat{w}_n' = \hat{u}_0 = 0$ at $z = \pm 1$.

4 Results

Weakly nonlinear analysis shows that the instability of the Hartmann flow is invariably subcritical regardless of the magnetic field strength [3]. In the present study, we determine how far the subcritical equilibrium states, which bifurcate from the Hartmann flow, extend below the linear stability threshold. Let us first validate our method described in Sec. 3.2 by computing critical Reynolds number for 2D travelling waves in plane Poiseuille flow, which corresponds to $Ha = 0$. To characterize the deviation of the velocity distribution from the base state, besides the transverse velocity amplitude A, we use also the amplitude associated with the energy of perturbation scaled by the energy of the basic flow $A_E^2 = \int_0^1 \langle|\vec{v}(x,z) - \vec{v}_0(z)|^2\rangle dz / \int_0^1 |\vec{v}_0(z)|^2 dz$, where the angular brackets denote for the streamwise average.

We start with a relatively low Hartmann number $Ha = 1$ for which the flow becomes linearly unstable at $Re_l = 10016.3$ with $\alpha_l = 0.971827$ [3]. The energy amplitude of equilibrium states versus the wavenumber is plotted in figure 1 for various subcritical values of Re. As for the non-magnetic plane Poiseuille flow, equilibrium states form closed contours, which shrink as Re is reduced, and collapse to a point at the critical $Re_n = 3961.36$ below which 2D travelling waves vanish. It means that subcritical perturbations have both a lower and an upper equilibrium amplitude. Both these amplitudes are plotted in figure 1 together with the respective value of $\hat{w}_1''(1)$, which is the quantity predicted by the weakly nonlinear analysis [3]. As seen, the the lower branch of $\hat{w}_1''(1)$ is predicted well by weakly nonlinear solution for subcritical Reynolds numbers down to $Re \approx 7000$.

A similar structure of subcritical equilibrium states is found also for $Ha = 5$ and $Re_n = 43\,8302$. At this large Re it becomes difficult to compute accurately the upper equilibrium states which require the numerical resolution of at least 48×32. The strongly subcritical states, which in this case extend down to $Re_n \approx 32860$, can reliably be computed with a substantially lower resolution of 48×10. In the following, we focus on such strongly subcritical Reynolds numbers at which 2D travelling waves emerge. The respective Reynolds number defines what is subsequently referred to as the 2D nonlinear stability threshold.

The critical Reynolds number and wavenumber for 2D nonlinear stability threshold are

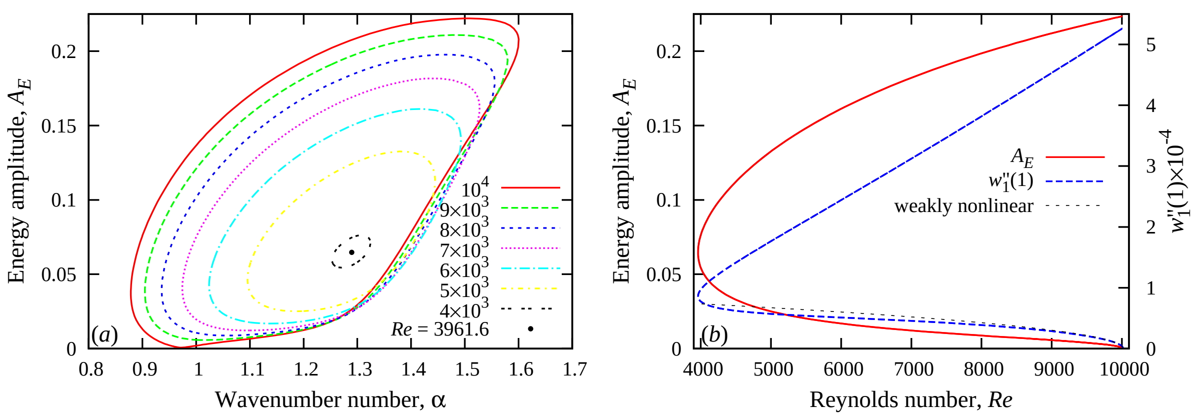

Figure 1: The energy amplitude of equilibrium states versus the wavenumber α for $Ha = 1$ and various Re computed with the resolution $M\times N = 32\times 8$.

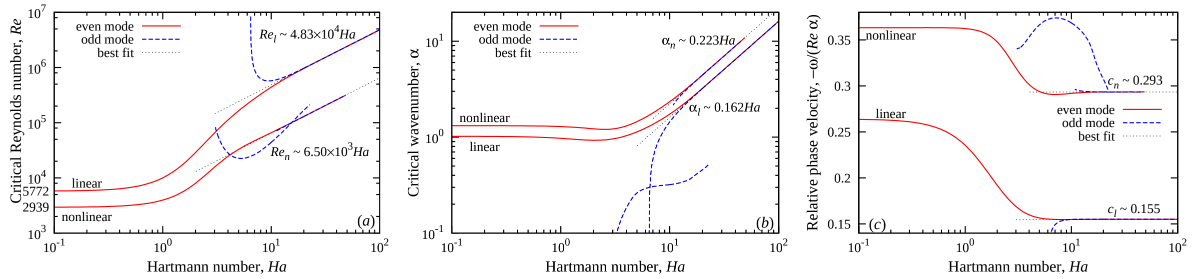

Figure 2: Critical Reynolds number (a), wavenumber (b) and phase speed (c) for even and odd modes of linear and nonlinear instabilities against the Hartmann number.

shown in figure 2 together with critical parameters for linear stability versus the Hartmann number [3]. At small Hartmann numbers, instability is associated with the even mode for which 2D travelling waves appear at $Re_n = 2939$. As the Hartmann number exceeds $Ha \approx 2.8$, which is about half of the respective value for the linear instability, an odd equilibrium mode appears with a large Reynolds number and a small wavenumber. This long-wave odd mode exists only within a limited range of Hartmann numbers up to $Ha \approx 20$. At $Ha \approx 10$ another odd mode appears with a slightly higher Reynolds number but much shorter wavelength. At $Ha \approx 15$ Reynolds number of the latter mode becomes smaller than that for the long-wave mode. The characteristics of this short-wave odd mode are seen closely approaching those of the original even mode. In a sufficiently strong magnetic field, the critical Reynolds number and wavenumber for both nonlinear modes increase with the Hartmann number similarly to the respective threshold parameters of the linear instability [3]. Namely, for $Ha \gtrsim 20$ the best fit yields $Re_n \sim 6.50 \times 10^3 Ha, \alpha_n \sim 0.223Ha, c_n \sim 0.293$. It is important to notice that the critical Reynolds number above is almost by an order of magnitude lower than that for the linear instability $Re_l \sim 48\,300Ha$. In the mean-field approximation using only one harmonic, we find $Re_n \sim 12\,300Ha$, which is almost by a factor of two higher than the accurate result above and coincides with the result reported by [5].

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