ON THE FLOW INSTABILITY
IN A HELICAL CHANNEL OF THE INDUCTION PUMP

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Abstract: A soft instability of helical flows of various structures arising under the impact of helically traveling magnetic field of the form $B_\rho = B_0 \cos(\omega t - k_\theta \theta)$ is examined. We consider inductionless approximation using the "external" friction model in the helical coordinate system ρ, θ, ζ . Neutral stability curves are calculated in logarithmic Hartmann-Reynolds plane for the pitch angle of helical channel $\alpha = 5^0$.

1. Introduction

The structure of flows arising in helical channels of induction pumps under the action of a rotating magnetic field (RMF) depends both on such characteristics of helical channels as curvature and torsion and on MHD flow parameters. A joint impact of these characteristics determines the channels drag and, to a certain extent, the efficiency of the pump operation.

2. Mathematical model

We examine soft instability of laminar and turbulent flows with respect to the appearance of stationary or non-stationary spatially periodic secondary structures or waves in the induction-free approximation and using the "external" friction model.

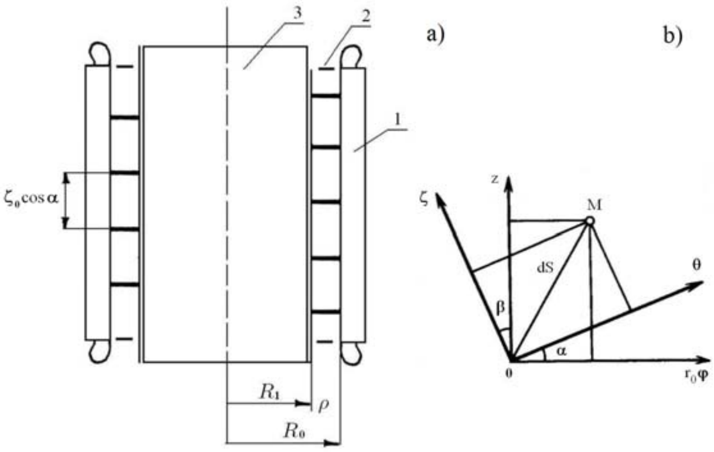

Figure1: Schematic diagram for estimations (a):1- RMF inductor; 2 - helical channel; 3 - ferromagnetic core; (b) – relations between helical and cylindrical coordinates.

MHD processes in helical channels are described by the following dimensionless vector equations:

$$\mathrm{Re}_\omega \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v}\nabla)\vec{v} \right] = -\mathrm{Re}_\omega \nabla P + L\vec{v} - \lambda \vec{v} + < \vec{f} >_{em}, \quad (1)$$

$$div\, \vec{v} = 0, \quad (2)$$

and boundary conditions $\vec{v}\big|_r = 0,$ (3)

where $\mathrm{Re}_\omega = \omega(R_0 - R_1)^2/\nu$, $Ha = B_0(R_0 - R_1)\sqrt{\sigma/\eta}$, $L = -rot\,rot$, $\vec{e}_i$ are the unit vectors of the helical coordinate system (HCS) ρ_0, θ, ζ (fig. 1) , $<\vec{f}>_{em} = \sum_{i=1}^{3} \vec{e}_i f_i$,

$<f_0> = Ha^2 Sk_{att}\varphi(\rho_0)\cos^2\alpha$; $S = 1 - \frac{v_\theta(r_1 + 0.5)\cos^2\alpha}{\rho_0}$; $\varphi(\rho_0) = \left(G_1\rho_0^{\frac{\mu-2}{2}} + G_2\rho_0^{-\frac{\mu+2}{2}}\right)^2$;

$\rho_0 = r_1 + \rho$; $\mu = \sqrt{1 + 4\cos^4\alpha}$; G_1, G_2 are functions of μ, r_1 [1], k_{att} is an attenuation factor [2], $\lambda = \frac{C_\varepsilon(\mathrm{Re}_\omega < V_0 >)^{1-\varepsilon}}{\delta_\zeta}$ is the "external" friction factor [3], $C_\varepsilon = 0.219\,\mathrm{e}^{10.895\varepsilon}$; $<V_f> = 0.5\omega(R_1 + R_0)$ mean velocity of magnetic field. Factor C_ε and structural parameter ε are determined using P-Q characteristic of the helical induction pump ACH-3 [2], $\delta_\zeta = \zeta_0/(R_0 - R_1)$. In case of the laminar flow $\lambda = 0$.

The system (1) - (3) components in HCS have the following form:

$$\mathrm{Re}_\omega\left[\frac{\partial v_\rho}{\partial t} + (v_i\cdot\nabla)v_j - \frac{v_\theta^2 + v_\zeta^2}{\rho_0}\right] = -\mathrm{Re}_\omega\frac{\partial P}{\partial \rho} + (\Delta - \lambda)v_\rho + f_\rho -$$
$$-\frac{2}{\rho_0^2}\left(\frac{\cos^2\alpha}{\delta_\theta}\frac{\partial v_\theta}{\partial\theta} + \frac{\sin^2\alpha}{\delta_\zeta}\frac{\partial v_\zeta}{\partial\zeta}\right),$$
$$\mathrm{Re}_\omega\left[\frac{\partial v_\theta}{\partial t} + (v_i\nabla)v_j + \frac{v_\rho v_\vartheta}{\rho_0}\right] = -\mathrm{Re}_\omega\frac{\cos^2\alpha}{\rho_0\delta_\theta}\frac{\partial P}{\partial\theta} + (\Delta - \lambda)v_\theta + f_\theta + \frac{2\cos^2\alpha}{\delta_\theta\rho_0^2}\frac{\partial v_\rho}{\partial\theta}, \quad (4)$$
$$\mathrm{Re}_\omega\left[\frac{\partial v_\zeta}{\partial t} + (v_i\nabla)v_j + \frac{v_\rho v_\zeta}{\rho_0}\right] = -\mathrm{Re}_\omega\frac{\sin^2\alpha}{\rho_0\delta_\zeta}\frac{\partial P}{\partial\zeta} + (\Delta - \lambda)v_\zeta + \frac{2\sin^2\alpha}{\rho_0^2\delta_\zeta}\frac{\partial v_\rho}{\partial\zeta} + f_\zeta,$$

where $(v_i\cdot\nabla)v_j = v_\rho\cdot\frac{\partial v_j}{\partial\rho} + \frac{v_\theta\cos^2\alpha}{\delta_\theta\rho_0}\frac{\partial v_j}{\partial\theta} + \frac{v_\zeta\sin^2\alpha}{\delta_\zeta\rho_0}\frac{\partial v_j}{\partial\zeta}$,

$f_\zeta = -2Ha^2 v_\zeta, < f_\theta > = f_0 - 2Ha^2 v_\theta$

$$\Delta v_i = \frac{\partial^2 v_i}{\partial\rho^2} + \frac{2}{\rho_0}\frac{\partial v_i}{\partial\rho} + \frac{\cos^4\alpha}{\delta_\theta^2\rho_0^2}\frac{\partial^2 v_i}{\partial\theta^2} + \frac{\sin^4\alpha}{\delta_\zeta^2\rho_0^2}\frac{\partial^2 v_i}{\partial\zeta^2}, \quad i, j = \rho, \theta, \zeta, \quad (5)$$
$$div\,\vec{v} = \frac{\partial v_\rho}{\partial\rho} + \frac{2}{\rho_0}v_\rho + \frac{\cos^2\alpha}{\delta_\theta\rho_0}\frac{\partial v_\theta}{\partial\theta} + \frac{\sin^2\alpha}{\delta_\zeta\rho_0}\frac{\partial v_\zeta}{\partial\zeta}.$$

3. Problem solution

The velocity of a unidirectional laminar or mean turbulent flow in a helical channel arising under the action of the force f_0 is determined from the solution of the following equation:

$$V_0'' + \frac{2}{\rho_0}V_0' - \lambda V_0 - Ha^2\cos^4\alpha\cdot\varphi(\rho_0)\frac{k_{att}(r_1 + 0.5)}{\rho_0}V_0 = -Ha^2\cos^4\alpha\cdot\varphi(\rho_0)k_{att} \quad (6)$$

with boundary conditions $V_0\big|_{\rho_0 = r_1} = V_0\big|_{\rho_0 = r_1 + 1} = 0.$

Following the Galerkin method, we obtain: $V_0 = \sum_{k=1}^{\infty} H_k\xi(\alpha_k\rho_0)$, (7)

where H_k is determined from the solution of k equations $\left\|H_k O_{kl} = P_l\right\|$,

$P_l = Ha^2\cos^2\alpha\cdot k_{att}In_{04l}$, $Q_{kl} = \alpha_k^2 In_{0kl} + \lambda In_{01kl} - 2\alpha_k In_{02kl} + Ha^2\cos^4\alpha\cdot k_{att}(r_1+0.5)In_{03kl}$,

$$\xi(\alpha_k\rho_0) = \sin\alpha_k\rho_0 + A_k sh\alpha_k\rho_0,\quad A_k = -\sin\alpha_k r_1 / sh\alpha_k r_1,;$$

α_k are the roots of the equation $\sin\alpha_k(r_1+1)sh\alpha_k r_1 - \sin\alpha_k r_1 sh\alpha_k(r_1+1) = 0.$

We specify small velocity and pressure disturbances in the form of waves travelling along the θ and ζ axes, whose amplitude depends on the coordinate ρ, i.e.

$$\begin{vmatrix} v_\rho \\ v_\theta \\ v_\zeta \\ P \end{vmatrix} = \begin{matrix} \\ V_0 + \\ \\ P_0 + \end{matrix}\begin{vmatrix} u(\rho) \\ v(\rho) \\ w(\rho) \\ q(\rho) \end{vmatrix}\times\exp[\sigma\tau + i(a_\theta\theta + a_\zeta\zeta)], \qquad (8)$$

where $\sigma = \sigma_R + i\sigma_I$.

Substituting (8) into (4), (5) and neglecting the squares of minor disturbances, we obtain the following system of equations connecting velocity and pressure disturbances:

$$gu - \frac{2\,\mathrm{Re}_\omega V_0}{\rho_0}v = -\mathrm{Re}_\omega Dq + \Delta u - \frac{2i}{\rho_0^2}(h_\theta v + h_\zeta w), \qquad (9)$$

$$gv + \mathrm{Re}_\omega(D_* V_0)u = -\frac{i\,\mathrm{Re}_\omega h_\theta}{\rho_0}q + \Delta v + \frac{2ih_\theta}{\rho_0^2}u, \qquad (10)$$

$$(g + Ha^2)w = -\frac{i\,\mathrm{Re}_\omega h_\zeta}{\rho_0}q + \Delta w + \frac{2ih_\zeta}{\rho_0^2}u, \qquad (11)$$

$$Du + \frac{2_0}{\rho_0}u + \frac{i}{\rho_0}(h_\theta v + h_\zeta w) = 0, \qquad (12)$$

where $g = \mathrm{Re}_\omega\sigma + \lambda + Ha^2$, $g_H = g + Ha^2$, $D + \dfrac{\partial}{\partial\rho}$, $D_* = \dfrac{\partial}{\partial\rho} + \dfrac{1}{\rho_0}$, $\sigma = \sigma_R + i\sigma_I$,

$\Delta = D^2 + \dfrac{2}{\rho_0}D - \dfrac{h^2}{\rho_0^2}$, $h^2 = h_\theta^2 + h_\zeta^2$, a_θ, a_ζ are components of the wave vector, $\delta_\theta = \theta_0/(R_0 - R_1)$, $\delta_\zeta = \zeta_0/(R_0 - R_1)$, $\theta_0 = \dfrac{\pi(R_0 + R_1)}{\cos\alpha}$ is the length of the channel turn, ζ_0 is the channel height, $h_\theta = \dfrac{a_\theta\cos^2\alpha}{\delta_\theta}$, $h_\zeta = \dfrac{a_\zeta\sin^2\alpha}{\delta_\zeta}$.

Expressing *w* through *u* and *v* using (12), determining *q* and *Dq* using (11) and excluding *q* from (9), (10), we obtain:

$$D(A_1 u) - h_\zeta^2(\Delta_1 - g)u + ih_\theta D(A_2 v) - \frac{2\,\mathrm{Re}_\omega h_\zeta^2 V_0}{\rho_0}v = 0, \qquad (13)$$

$$\left[\frac{h_\theta^2}{\rho_0}\left(A_1 - 2ih_\zeta^2\right) + \mathrm{Re}_\omega h_\zeta^2(D_* V_0)\right]u + \left[\frac{ih_\theta^2}{\rho_0}A_2 - h_\zeta^2(\Delta + g)\right]v = 0, \qquad (14)$$

where A_1, A_2, Δ_1 are linear differential operators:

$$A_1 = \rho_0^2 D^3 + 6\rho_0 D^2 + (6 - h^2 - g_H\rho_0^2)D - 2(h^2/\rho_0^2 + g_H\rho_0),$$

$$A_2 = \rho_0 D^2 + 2D - (h^2/\rho_0 + g_H\rho_0^2), \quad \Delta_1 = D^2 + \frac{4}{\rho_0}D + \frac{4 - h^2}{\rho_0^2}.$$

Further, the problem of the stability of laminar or mean turbulent flow profile is considered within the sub-region $0 \le \rho \le 1$ of the range of values ρ_0. In this case, boundary conditions for the system (13), (14) are

$$u\big|_{\rho=0} = u\big|_{\rho=1} = Du\big|_{\rho=1} = 0, \qquad (15)$$

$$v\big|_{\rho=0} = v\big|_{\rho=1} = 0. \qquad (16)$$

We seek the solution to Eqs. (13)-(14) by the Galerkin method in the form

$$u = \sum_{k=1}^{\infty} C_k \zeta(\gamma_k \rho), \qquad (17)$$

$$v = \sum_{k=1}^{\infty} B_k k\pi\rho, \qquad (18)$$

where C_k, B_k are complex coefficients of the expansions (17), (18),

$$\zeta(\gamma_k \rho) = \sin\gamma_k \rho + A_k^* sh\gamma_k \rho, \qquad (19)$$

$A_k^* = -\sin\gamma_k / sh\gamma_k$, γ_k are roots of the equation $\sin\gamma_k ch\gamma_k - sh\gamma_k \cos\gamma_k = 0.$

Substituting the series (17), (18) into Eqs. (13), (14) and accomplishing the procedure of Galerkin's method, we obtain a system of 2*l* homogeneous algebraic equations binding the coefficients C_k and B_k. Using the solvability condition for this system, singling out the real and imaginary parts of the obtained expression, using, to the first approximation, the stability change principle $\sigma = 0$ and assuming that $h_\theta = 0, k = l = 1,$ we obtain an equation connecting critical values of MHD parameters Ha, Re_ω and a_ζ:

$$\begin{aligned}&\{\gamma_k^4 In_{1kl} - 8\gamma_k^3 In_{2kl} - \gamma_k^2[(12-h_\zeta^2)In_{3kl} - g_H In_{4kl}] - 4\gamma_k(g_H In_{5kl} + h_\zeta^2 In_{6kl}) - \\ &-2(g_H In_{7kl} - h_\zeta^2 In_{8kl}) + h_\zeta^2[\gamma_k^2 In_{9kl} + 4\gamma_k In_{10kl} + (4-h_\zeta^2)In_{11kl} + g_R In_{12kl}]\}\times \\ &\times(k^2\pi^2 In_{13kl} - 2k\pi In_{14kl} + h_\zeta^2 In_{15kl} + g_R In_{16kl}) - 2\,\mathrm{Re}_\omega^2 h_\zeta^2 In_{17kl} In_{18kl} = 0,\end{aligned} \qquad (20)$$

where the integrals In_{kl} are given in the Appendix.

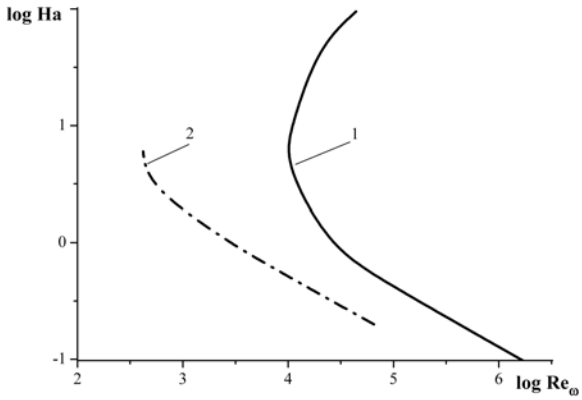

Figure 2: Estimations of neutral stability parameters for MHD flows: 1 – in helical channel, 2 – in cylindrical cavity [4].

Our estimations by (20) show (fig.2) that helical MHD flows under study possess of higher stability level relatively to disturbances of Taylor vortices type in comparison with rotational MHD flows in cylindrical cavities under RMF effect [4].

4. Conclusion

A complete solution of the instability problem of MHD flows in helical channels using the "external" friction model was derived. Neutral stability conditions were analyzed in the first approximation at $\sigma = 0,\ \ k = l = 1$. Obtained estimations were compared with known ones for rotating MHD flows in cylindrical channels under the impact of RMF.

5. References

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Appendix

The following are the integrals used in the problem (*k, l* integral indices are not shown for simplicity):

$$In_0 = \int_{r_1}^{r_1+1} \xi^*(\alpha_k\rho_0)\xi(\alpha_l\rho_0)d\rho,\ \ In_{01} = \int_{r_1}^{r_1+1} \xi(\alpha_k\rho_0)\xi(\alpha_l\rho_0)d\rho,$$

$$In_{02} = \int_{r_1}^{r_1+1} \rho_0^{-1}(\cos\alpha_k\rho_0 + A_k\ ch\alpha_k\rho_0)\zeta(\gamma_l\rho_0)d\rho,\ In_{03} = \int_{r_1}^{r_1+1} \rho_0^{-1}\varphi(\rho_0)\xi(\alpha_k\rho_0)\xi(\alpha_l\rho_0)d\rho,$$

$$In_{04} = \int_{r_1}^{r_1+1} \varphi(\rho_0)\xi(\alpha_l\rho_0)d\rho;$$

$$In_1 = \int_0^1 \rho_0^2\zeta(\gamma_k\rho)\zeta(\gamma_l\rho)d\rho, \qquad In_2 = \int_0^1 \rho_0(\cos\gamma_k\rho - A_k^* ch\,\gamma_k\rho)\zeta(\gamma_l\rho)d\rho,$$

$$In_3 = \int_0^1 (\sin\gamma_k\rho - A_k^* sh\,\gamma_k\rho)\zeta(\gamma_l\rho)d\rho, \qquad In_4 = \int_0^1 \rho_0^2(\sin\gamma_k\rho - A_k^* sh\,\gamma_k\rho)\zeta(\gamma_l\rho)d\rho,$$

$$In_5 = \int_0^1 \rho_0(\cos\gamma_k\rho + A_k^* ch\,\gamma_k\rho)\zeta(\gamma_l\rho)d\rho, \quad In_6 = \int_0^1 \rho_0^{-1}(\cos\gamma_k\rho + A_k^* ch\,\gamma_k\rho)\zeta(\gamma_l\rho)d\rho,$$

$$In_7 = \int_0^1 (\cos\gamma_k\rho + A_k^* ch\,\gamma_k\rho)\zeta(\gamma_l\rho)d\rho, \quad In_8 = \int_0^1 \rho_0^{-2}\zeta(\gamma_k\rho)\zeta(\gamma_l\rho)d\rho, \quad In_9 = In_3,$$

$$In_{10} = In_6, \quad In_{11} = In_8, \quad In_{12} = \int_0^1 \zeta(\gamma_k\rho)\zeta(\gamma_l\rho)d\rho, \quad In_{13} = \int_0^1 \sin k\pi\rho \sin l\pi\rho d\rho,$$

$$In_{14} = \int_0^1 (\cos\gamma_k\rho + A_k^* ch\,\gamma_k\rho)\sin l\pi\rho d\rho, \quad In_{15} = \int_0^1 \rho_0^{-2}\sin k\pi\rho \sin l\pi\rho d\rho,$$

$$In_{16} = \int_0^1 (\sin\gamma_k\rho - A_k^* sh\,\gamma_k\rho)\sin l\pi\rho d\rho, \quad In_{17} = \int_0^1 \rho_0^{-1}V_0 \sin k\pi\rho\zeta(\gamma_l\rho)d\rho,$$

$$In_{18} = \int_0^1 (D_* V_0)\zeta(\gamma_k\rho)\sin l\pi\rho d\rho.$$