Stability investigation of Hartmann flow with the convective approximation

I.Yu. Kalashnikov

National Research Nuclear University MEPhI, department 32,
Kashira highway, 31.

kalasxel@gmail.com

Abstract

The report focuses on the linear analysis of a plane-parallel flow stability in transverse magnetic field (Hartmann flow) with the convective approximation. Obtained and solved equations that describes the perturbation growth. Founded perturbation modes and non-excitation conditions of ones. Obtained the equation for the instability increment and shown to have an instable root of the equation. Also shown that founded instabilities is qualitatively agrees with the experimental data.

Introduction

Hartmann flow is steady stream between two fixed infinite parallel planes arising for due to the pressure drop that occurs in a magnetic field directed perperpendicular planes. We choose the z axis co-directional with the external magnetic field B_0, and the x axis direct along the stream. For such flow there is an exact solution:

$$V_x(z) = \frac{k_2\delta}{k_1 \sinh(k_1\delta)}(\cosh(k_1\delta) - \cosh(k_2 z)), \tag{1}$$

$$\sqrt{\frac{\nu_m}{4\pi\rho\nu}} B_x = -\frac{k_2}{k_1} z + \frac{k_2\delta}{k_1 \sinh(k_1\delta)} \sinh(k_1 z), \tag{2}$$

where $k_1 = B_0/\sqrt{4\pi\rho\nu\nu_m}$, $k_2 = -(1/\rho\nu)(\partial p/\partial x)$. Constants ν and ν_m are kinematic and magnetic viscosity, ρ is density of the fluid.

From (1), (2) we can show that with increasing transverse magnetic fields velocity profile becomes flatter. This flatness is characterized by the Hartmann number: $Ha = B_0/\sqrt{4\pi\rho\nu\nu_m}$.

Stability of the Hartmann flow was first considered in [1] where the influence of the magnetic field is taken into account only by changing the velocity field, and obtained (quite expected) result that with increasing magnetic field stability increases too.

Also worth noting the work of [2], which has been studied experimentally the transition to turbulence due to the instability Hartmann layer and conditions of turbulence suppression. Found that when the parameter $R = Re/Ha > 380$ the flow becomes turbulent. Numerical simulations [3] gives approximately the same result. Because R is inversely proportional to the magnetic field, then again, it can be concluded that a weak magnetic field destabilizes the current.

Two-dimensional perturbations

Assume that instabilities are convective, ie perturbations that arise at any point does not have time to develop, and are carried over beyond the real pipe. But because the magnetic field has (because embeddedness) inhibitory effect, then feasibility of such an assumption, it should be small.

We can leave this value from the dimensional parameters of the liquid in three ways: ν_m/δ, ν/δ and $\sqrt{\nu\nu_m}/\delta$. Since on embeddedness affects only magnetic viscosity, it is logical to choose the first variant. Thus we obtain:

$$\frac{B_0\delta}{\sqrt{4\pi\rho\nu_m}} \ll 1. \tag{3}$$

In this approximation, we consider that the perturbation does not evolve, moving along the main flow, ie $\partial/\partial x = 0$. Then investigate the stability of the system of equations:

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}, \nabla)\mathbf{V} + \frac{1}{\rho}(P + \frac{\mathbf{B}^2}{8\pi}) - \frac{1}{4\pi\rho}(\mathbf{B}, \nabla)\mathbf{B} - \nu\nabla^2\mathbf{V} = 0, \tag{4}$$

$$\frac{\partial \mathbf{B}}{\partial t} - (\mathbf{B}, \nabla)\mathbf{V} + (\mathbf{V}, \nabla)\mathbf{B} - \nu_m\nabla^2\mathbf{V} = 0, \tag{5}$$

$$\nabla \cdot \mathbf{V} = 0, \tag{6}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{7}$$

assuming that the main flow is obeys (1), (2), and all the convective instability.

Making the transformation $\mathbf{B} \to \mathbf{B} + \mathbf{b}$, $\mathbf{V} \to \mathbf{V} + \mathbf{v}$, $P \to P + \varphi$, where $\mathbf{V} = (V(z), 0, 0)$, $\mathbf{B} = (B(z), 0, B_0)$ - known. We leave only linear members of perturbations. Further, since on the time t and the coordinate y movement is infinite, then assume that on these variables perturbation is periodic : $f(y, z, t) \to f(z)exp(i\gamma t - iky)$. Ie in each layer dz extends plane wave. Denoting then $b_x = b$ and $v_x = v$ write:

$$\mathbf{b} = (b, \frac{\partial a}{\partial z}, -\frac{\partial a}{\partial y}) = (b, \frac{\partial a}{\partial z}, ika), \tag{8}$$

$$\mathbf{v} = (v, \frac{\partial q}{\partial z}, -\frac{\partial q}{\partial y}) = (v, \frac{\partial q}{\partial z}, ikq), \tag{9}$$

where a and q - x components of the vector potential.

Through this transformation of equation (6) and (7) disappears, and y and z components of the equation (5) are identical. Then the system equations describing the amplitude of a plane wave perturbation has the form:

$$i\gamma v + ikV'q - \frac{B_0}{4\pi\rho}\frac{db}{dz} - ik\frac{B'}{4\pi\rho}a + \nu k^2 v - \nu\frac{d^2b}{dz^2} = 0, \tag{10}$$

$$i\gamma b - ikV'a - B_0\frac{dv}{dz} + ikB'q + \nu_m k^2 b - \nu_m\frac{d^2b}{dz^2} = 0, \tag{11}$$

$$\frac{dM}{dz} - ikN = \frac{B_0}{4\pi\rho}\nabla^2 a, \tag{12}$$

$$\frac{dN}{dz} + ikM = 0, \tag{13}$$

where stood the two quantities:

$$M = i\gamma q + \nu k^2 q - \nu \frac{d^2 q}{dz^2}, \tag{14}$$

$$N = \frac{\varphi}{\rho} + \frac{Bb}{4\pi\rho}. \tag{15}$$

For N values the boundary conditions are obviously zero: $N(\pm\delta) = 0$. However, you need another condition to determine the possible wave numbers k. Note that if in the expression for M make replacement $z \to -z$ (or $k \to -k$) it does not change. So we can say that $M(\delta) = M(-\delta)$.

Then from (14), (15) and substituted the boundary conditions we obtain the eigenvalues of the wave number:

$$k = -i\frac{\pi}{\delta}n, \quad n \in \mathbb{Z}. \tag{16}$$

Ie for $k = 0$, there is instability, being increased either the right or left relative to the flow. Nonzero modes will not be excited, if δ is sufficiently large. We can compare δ with the parameters of liquid three ways, but since this instability is due to hydrodynamic and electrodynamic forces then choose the variant for δ, where ν and ν_m includes equally. Therefore, considering convection assumptions we have range for δ:

$$\frac{\sqrt{4\pi\rho\nu\nu_m}}{B_0} \ll \delta \ll \frac{\sqrt{4\pi\rho}\nu_m}{B_0}, \tag{17}$$

which implies that $\nu \ll \nu_m$ or magnetic Prandtl number $Pr_m \ll 1$. And as $Ha \gg 1$.

One-dimensional perturbations

It should be noted that for $k \neq 0$ system obtained can be solved exactly, but we restrict our investigation is one-dimensional flow in the y-stable region. Then the vector perturbations are two-dimensional (no z component). We obtain two independent systems for the potentials (18), (19) and for the components of the perturbation (20), (21):

$$i\gamma a - B_0 \frac{dq}{dz} - \nu_m \frac{d^2 a}{dz^2} = 0, \tag{18}$$

$$i\gamma q - \frac{B_0}{4\pi\rho}\frac{da}{dz} - \nu \frac{d^2 q}{dz^2} = 0, \tag{19}$$

$$i\gamma b - B_0 \frac{dv}{dz} - \nu_m \frac{d^2 b}{dz^2} = 0, \tag{20}$$

$$i\gamma v - \frac{B_0}{4\pi\rho}\frac{db}{dz} - \nu \frac{d^2 v}{dz^2} = 0, \tag{21}$$

the pressure disturbance Is expressed through disturbance of the magnetic field as follows:

$$\varphi = -\frac{Bb}{4\pi}. \tag{22}$$

Boundary conditions are as follows:

$$b(\pm\delta) = v(\pm\delta) = \frac{da}{dz}(\pm\delta) = \frac{dq}{dz}(\pm\delta) = 0. \qquad (23)$$

Note that when $\nu = \nu_m$ in both systems is the symmetry: when replacing $b \to v\sqrt{4\pi\rho}$ (or $a \to q\sqrt{4\pi\rho}$) form of the equations is not changed. Therefore, the spectrum obtained is a degenerate, so that the eigenfunctions can be found in the form $b = \sigma v\sqrt{4\pi\rho}$, where $\sigma = \pm 1$. We have eigenvalues corresponding to stable flow:

$$i\gamma = -\frac{\nu}{(2\delta)^2}(\pi^2 n^2 + Ha^2) \leq 0, \;\; n \in \mathbb{Z}. \qquad (24)$$

This means that for close values of kinematic and magnetic viscosity low and one-dimensional perturbations are damped and the flow is stable. However, as follows from (17), there already been undamped perturbation mode.

System (18), (19) and (20), (21) in the solution will give the same eigenvalues λ and lead to the same equation for the eigenvalues of the increment γ. We solve (20), (21). We seek a solution in the form $b = b_0 e^{\lambda z}$, $v = v_0 e^{\lambda z}$. Obtain the eigenvalues:

$$\lambda^2 = \frac{1}{2}\left[i\gamma\left(\frac{1}{\nu} + \frac{1}{\nu_m}\right) + \frac{{B_0}^2}{4\pi\rho\nu\nu_m} \pm \sqrt{\left[i\gamma\left(\frac{1}{\nu} + \frac{1}{\nu_m}\right) + \frac{{B_0}^2}{4\pi\rho\nu\nu_m}\right]^2 - \frac{4\gamma^2}{\nu\nu_m}}\,\right]. \qquad (25)$$

In view of the boundary conditions we obtain the equation for increment γ:

$$b_1 v_2 \cosh(\lambda_1\delta)\sinh(\lambda_2\delta) = v_1 b_2 \sinh(\lambda_1\delta)\cosh(\lambda_2\delta), \qquad (26)$$

where: $b_1 = B_0\lambda_1\delta$, $b_2 = B_0\lambda_2\delta$, $v_1 = \delta(i\gamma - \nu_m{\lambda_1}^2)$, $v_2 = \delta(i\gamma - \nu_m{\lambda_2}^2)$ (λ_1 taken with plus before the square root).

Dimensionless $\gamma\delta^2/\nu \to \gamma$. In the first approximation we set $Pr_m = 0$. Then we obtain the following for the eigenvalues: $\lambda_1\delta \simeq \sqrt{i\gamma + Ha^2}$, $\lambda_2 \simeq 0$. Equation (26) can be factored and written in the form:

$$i\gamma\sqrt{i\gamma + Ha^2} = \tanh\sqrt{i\gamma + Ha^2}. \qquad (27)$$

Roots of equation (27) must satisfy equation (26) in the approximation $Pr_m = 0$.

One of the roots (stable) is immediately visible: $i\gamma = -Ha^2 < 0$. Another root corresponding a pure imaginary increment indicates flow instability: $i\gamma > 0$.

Discussion and conclusions

On Fig. 1 shows a summary of the experimental results on the study of the stability of the Hartmann flow [4]. In these experiments, the measured resistance coefficient: $\lambda = -2p'\delta/\rho V^2$. For Hartmann flow it has the form: $\lambda_H \simeq 2Ha/Re$. At the graph the deviation of bisector from the coordinate angle means that flow for the given parameters already turbulent.

From (3) follow that in the convective approximation $\lambda \ll Ha^{-1}{Re_m}^{-1}$. Since the experiments were carried out at $Ha \sim 10^2$ and $Re_m \sim 10^{-3}$ (mercury) it is possible to say that

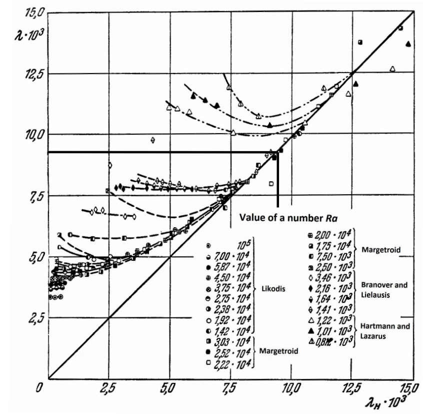

Figure 1: Experimental data on the resistance coefficient in comparison with the theory for the flow of Hartmann. Highlighted the range of applicability of the convective approximation.

$\lambda_H \ll 10$. And as can be seen from Fig. 1 in this field flow is not laminar. Thus found instability really take place.

Fulfillment of the condition (17) provides only the absence of perturbations aimed parallel to the planes (ie absence component of the perturbation).As seen from (27) with increasing Ha instability is suppressed. This can be explained by the fact that with increasing magnetic field (and as a consequence an increase in the Hartmann number) embeddedness effect will dominate, making it difficult to form instabilities in the initially laminar flow instabilities.But the equation (27) obtained in the limit $Pr_m \to 0$. To determine more precise conditions under which the flow could be sustained needs detailed analysis of (26).

It should also be noted that in the limit $Ha \to 0$ instability is still present, although it itself becomes for Poiseuille. Absence of such a limit transition observed in [5], and explained that a non-zero magnetic field already generates instability, which then can develop without using a magnetic field.

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