The effect of axial electric current on the helical magneterotational instability

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Abstract: We present the results of numerical stability analysis of a cylindrical Taylor-Couette flow of liquid metal carrying an axial electric current in the presence of a generally helical external magnetic field. Two purely electromagnetic instabilities are found in the presence of the electric current. The first is a pinch-type instability driven by the interaction of electric current with its own magnetic field, which is known as the Tayler instability. The axisymmetric mode of this instability requires a free-space component of the azimuthal magnetic field which is possible in annular but in solid cylindrical geometry. The second appears to be a new type of electromagnetic instability driven by the interaction of electric current with a weak collinear external magnetic field.

1 Introduction

The present paper is concerned with the effect of the axial current passing through liquid metal in the Taylor-Couette set-up on the so-called helical magnetorotational instability (HMRI). However the HMRI is able to destabilize centrifugally stable velocity distributions [1, 2], it does not reach up to the astrophysically relevant Keplerian rotation profile [3, 4]. Recently, it has been suggested that this limitation of HMRI can be overcome when the azimuthal magnetic field component is allowed to have a non-zero rotation, which means an electric current passing through the fluid [5]. From the physical point of view, current provides an additional energy source. Thus, instability no longer requires background flow and so can extend over an unlimited range of velocity profiles. In this paper, we show that there are two such instabilities which appear in the presence of a background electric current. The first is the so-called Tayler instability which is a pinch-type instability driven by the interaction of electric current with its own magnetic field [6]. The second is a new type of instability driven by the interaction of electric current with a weak collinear external magnetic field.

2 Formulation of the problem

Consider an incompressible fluid of kinematic viscosity ν and electrical conductivity σ filling the gap between two infinite concentric cylinders with inner radius R_i and outer radius R_o rotating respectively with angular velocities Ω_i and Ω_o in the presence of helical magnetic field $\vec{B}_0 = \vec{e}_z B_z + \vec{e}_\phi B_\phi$ with the axial component $B_z = \alpha B_0$ and the azimuthal component

$$B_\phi = B_0 \left[(\beta - \gamma) R_i / r + \gamma r / R_i \right] \tag{1}$$

in cylindrical coordinates (r, ϕ, z). The dimensionless coefficient α defines the magnitude of the axial component of the magnetic field relative to that of the azimuthal component. The latter has a free-space part defined by the coefficient β and a rotational part defined by the coefficient γ which is associated with the axial current density in the fluid $\vec{j}_0 = \frac{1}{\mu_0} \vec{\nabla} \times \vec{B}_0 = \vec{e}_z \frac{2\gamma B_0}{\mu_0 R_i}$, where μ_0

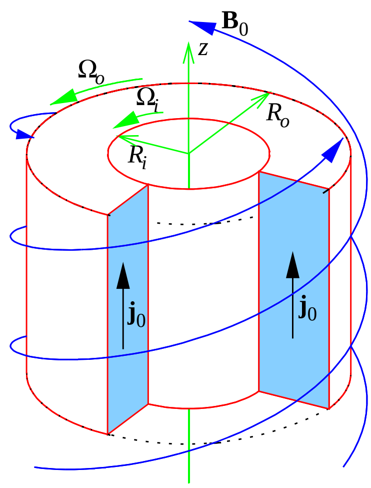

Figure 1: Sketch of the problem.

is the magnetic permeability of vacuum. Note that in annular geometry with $R_i \neq 0$, the latter produces also a free-space component with the effective helicity $-\gamma$.

Following the inductionless approximation, which holds for most of liquid-metal magnetohydrodynamics characterized by small magnetic Reynolds numbers $Re_m = \mu_0\sigma v_0 L \ll 1$, where v_0 and L are the characteristic velocity and length scales, the magnetic field of the currents induced by the fluid flow is assumed to be negligible relative to the imposed field $\vec{B}_0$ everywhere except the Navier-Stokes equation

$$\partial_t \vec{v} + (\vec{v}\cdot\vec{\nabla})\vec{v} = \rho^{-1}\left(-\vec{\nabla}p + \vec{j}\times\vec{B}\right) + \nu\vec{\nabla}^2\vec{v}, \tag{2}$$

where, as shown below, its interaction with the background electric current $\vec{j}_0$ results in a non-negligible perturbation of the electromagnetic body force. The electric current is governed by Ohm's law for a moving medium $\vec{j} = \sigma\left(\vec{E} + \vec{v}\times\vec{B}_0\right)$ and related to the magnetic field by Ampère's law $\vec{j} = \mu_0^{-1}\vec{\nabla}\times\vec{B}$. In addition, we assume that the characteristic time of velocity variation is much longer than the magnetic diffusion time $\tau_0 \gg \tau_m = \mu_0\sigma L^2$. This leads to the quasi-stationary approximation, according to which $\vec{\nabla}\times\vec{E} = 0$ and $\vec{E} = -\vec{\nabla}\Phi$, where Φ is the electrostatic potential. Mass and charge conservation imply $\vec{\nabla}\cdot\vec{v} = \vec{\nabla}\cdot\vec{j} = 0$.

The problem admits base state with a purely azimuthal velocity distribution $\vec{v}_0(r) = \vec{e}_\phi v_0(r)$, where $v_0(r) = r\frac{\Omega_o R_o^2 - \Omega_i R_i^2}{R_o^2 - R_i^2} + \frac{1}{r}\frac{\Omega_o - \Omega_i}{R_o^{-2} - R_i^{-2}}$. Note that the magnetic field does not affect the base flow because it gives rise only to the electrostatic potential $\Phi_0(r) = B_0\int v_0(r)\,dr$ whose gradient compensates the induced electric field so that there is no current in the base state ($\vec{j}_0 = 0$). However, a current may appear in a perturbed state

$$\left\{\begin{array}{c}\vec{v},p\\ \vec{B},\Phi\end{array}\right\}(\vec{r},t) = \left\{\begin{array}{c}\vec{v}_0,p_0\\ \vec{B}_0,\Phi_0\end{array}\right\}(r) + \left\{\begin{array}{c}\vec{v}_1,p_1\\ \vec{B}_1,\Phi_1\end{array}\right\}(\vec{r},t),$$

where $\vec{v}_1$, p_1, $\vec{B}_1$, and Φ_1 present small-amplitude perturbations.

In the following, we focus on axisymmetric perturbations for which the solenoidity constraints are satisfied by meridional stream functions for fluid flow and electric current as $\vec{v} = v\vec{e}_\phi + \vec{\nabla}\times(\psi\vec{e}_\phi)$, $\vec{j} = j\vec{e}_\phi + \vec{\nabla}\times(h\vec{e}_\phi)$. Note that h is the azimuthal component of the induced magnetic field which is used subsequently instead of Φ for the description of the induced current. For numerical purposes, we introduce also the vorticity $\vec{\omega} = \omega\vec{e}_\phi + \vec{\nabla}\times(v\vec{e}_\phi) = \vec{\nabla}\times\vec{v}$ as an auxiliary variable. The perturbation is sought in the normal mode form $\{v_1, \omega_1, \psi_1, h_1, g_1\}(\vec{r},t) =$

$\{\hat{v}, \hat{\omega}, \hat{\psi}, \hat{h}, \hat{g}\}(r) \times e^{\gamma t+ikz}$, where γ is, in general, a complex growth rate and k is the axial wave number which is real for the conventional stability analysis and complex for absolute instability. Henceforth, we proceed to dimensionless variables by using R_i, R_i^2/ν, $R_i\Omega_i$, B_0, and $\sigma\mu_0 B_0 R_i^2 \Omega_i$ as the length, time, velocity, and the induced magnetic field scales, respectively. Non-dimensionalised governing equations then read as

$$\begin{aligned}
\gamma\hat{v} &= D_k\hat{v} + Re\, ikr^{-1}(r^2\Omega)'\hat{\psi} + Ha^2(ik\alpha\hat{h} + 2\gamma\hat{g}), && (3)\\
\gamma\hat{\omega} &= D_k\hat{\omega} + 2Re\, ik\Omega\hat{v} + Ha^2 ik[ik\alpha\hat{\psi} - 2((\beta-\gamma)r^{-2} + \gamma)\hat{h}], && (4)\\
0 &= D_k\hat{\psi} + \hat{\omega}, && (5)\\
0 &= D_k\hat{h} + ik[\alpha\hat{v} - 2(\beta-\gamma)r^{-2}\hat{\psi}], && (6)\\
0 &= D_k\hat{g} + k^2\alpha\hat{\psi}, && (7)
\end{aligned}$$

where $D_k f \equiv r^{-1}(rf')' - (r^{-2} + k^2)f$ and the prime stands for $\frac{\mathrm{d}}{\mathrm{d}r}$; $Re = R_i^2\Omega_i/\nu$ and $Ha = R_i B_0\sqrt{\frac{\sigma}{\rho\nu}}$ are Reynolds and Hartmann numbers, respectively; $\Omega(r) = \frac{\lambda^{-2}-\mu+r^{-2}(\mu-1)}{\lambda^{-2}-1}$ is the dimensionless angular velocity of the base flow defined by $\lambda = R_o/R_i$ and $\mu = \Omega_o/\Omega_i$.

The boundary conditions for the flow perturbation on the inner and outer cylinders at $r = 1$ and $r = \lambda$, respectively, are $\hat{v} = \hat{\psi} = \hat{\psi}' = 0$. The boundary conditions for $\hat{h}$ for insulating and perfectly conducting cylinders, respectively, are $\hat{h} = 0$ and $(r\hat{h})' = 0$ at $r = 1; \lambda$. The boundary conditions for the radial component of the induced magnetic field $\hat{g}$ follow from the solution of Eq. (7) in the free space, where $\hat{\psi} \equiv 0$. Thus, we have $\hat{g}(r) = G_i I_1(kr)$ and $\hat{g}(r) = G_o K_1(kr)$ for $0 \le r \le 1$ and $r \ge \lambda$, respectively, where I_n and K_n are the modified Bessel functions of the first and second types of index n. Taking the ratio $(r\hat{g})'/\hat{g}$ to eliminate the unknown constants G_i and G_o leads to the following boundary conditions for $\hat{g}$ $(r\hat{g})' = c_i(kr)\hat{g}$ at $r = 1$ and $(r\hat{g})' = c_o(kr)\hat{g}$ at $r = \lambda$, where $c_i(kr) = krI_0(kr)/I_1(kr)$ and $c_o(kr) = -krK_0(kr)/K_1(kr)$.

3 Results

In the following, the ratio of radii of inner and outer cylinders is fixed to $\lambda = 2$ and the cylinders are assumed to be insulating. We start with a hydrodynamically unstable flow corresponding to the ratio of rotation rates $\mu = 0.2$, which is below the Rayleigh limit $\mu_c = \lambda^{-2} = 0.25$. The magnetic field is helical with the axial component fixed by $\alpha = 1$ and the azimuthal component generated only by the current passing through the fluid which corresponds $\beta = 0$. In purely axial magnetic field corresponding to $\gamma = 0$, the flow becomes centrifugally unstable to stationary Taylor vortices when Reynolds number exceeds the marginal value which is plotted in Fig. 2 against the wave number. Addition of a weak azimuthal component of the magnetic field

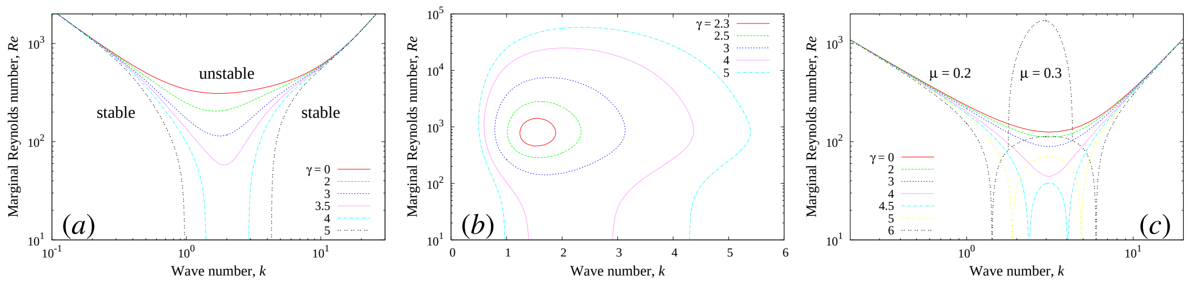

Figure 2: Marginal Reynolds number versus the wave number for a hydrodynamically unstable flow with $\mu = 0.2$ (a) and for a hydrodynamically stable flow with $\mu = 0.3$ (b) at various helicities γ of rotational helical magnetic field with $\alpha = 1$, $\beta = 0$ and $\alpha = \beta = 0$ (c) for $Ha = 10$.

reduces the instability threshold and makes the instability oscillatory ($\omega \neq 0$). However, the main result seen in Fig. 2(a) is the drop of the marginal Reynolds number to zero in a range of intermediate wave numbers when $\gamma \gtrsim 3.7$. Zero Reynolds number means that this instability is entirely electromagnetic and independent of the base flow. We will see later that there are two different electromagnetic mechanisms driving this instability.

Next, let us turn to the hydrodynamically stable flow with the ratio of rotation rates set to $\mu = 0.3 > \mu_c$ which is above the Rayleigh limit. As seen in Fig. 2(b), a moderately helical rotational magnetic field can destabilize this flow similarly to helical free-space magnetic field. In both cases, neutral stability curves from closed contours which means that the instability is constrained to certain ranges of the wave numbers and Reynolds numbers. Namely, in contrast to the hydrodynamically unstable case considered above, there are now two marginal Reynolds numbers – the lower one by exceeding which the flow destabilizes, and the upper one by exceeding which the flow restabilizes. This picture changes when the helicity of the rotational field exceeds$\gamma \approx 3.7$. As for the hydrodynamically unstable case considered above, marginal Reynolds number again drops to zero in a certain range of intermediate wave numbers when γ exceeds this critical value.

Let us consider what happens when the axial component of the magnetic field is switched off by setting $\alpha = 0$. It means that the magnetic field is purely azimuthal and it is generated only by the axial current passing in the liquid annulus. Marginal Reynolds number and the frequency for both hidrodynamically unstable ($\mu = 0.2$) and stable ($\mu = 0.3$) flows in the magnetic fields of various strength defined by γ and $Ha = 10$ is plotted in Fig. 2(c) against the wave number. In the hydrodynamically unstable case, the effect of purely azimuthal field is very similar to that of the helical field considered above. Namely, the increase of the axial electric current defined by γ results in the decrease of marginal Reynolds number, which again drops to zero in a certain range of wave numbers when $\gamma \gtrsim 4.5$. In contrast to helical magnetic field, now the instability is basically stationary ($\omega = 0$), although some oscillatory modes appear in the hydrodynamically stable case at high sufficiently high γ and Reynolds numbers. Also the hydrodynamically stable flow is affected by this purely azimuthal magnetic field in a slightly different way. Namely, all neutral stability curves in this case end at zero Reynolds number. Thus, the lower critical Reynolds number, if any, is always zero in the hydrodynamically case.

Obviously, here we have a z-pinch-type instability which occurs due to a weak compression of the azimuthal magnetic field lines by a radially inward meridional flow perturbation. This enhances the electromagnetic pinch force generated by the interaction of the axial electric current with its own magnetic field and, thus, amplifies the initial perturbation. It is important to notice that axisymmetric meridional flow affects only the free-space ($\sim r^{-1}$) but not the rotational ($\sim r$) component of the azimuthal magnetic field. The respective induction term is absent in Eq. (6) because axisymmetric meridional flow conserves the flux of the rotational azimuthal magnetic field. Thus, besides the rotational component this instability requires also a free-space component of the azimuthal magnetic field. The latter, however, is possible only in annular but not in cylindrical geometry. As seen from Eq. (1), the free-space component of the azimuthal magnetic field associated the axial electric current in annular geometry ($R_i \neq 0$) can be compensated by additional free-space magnetic field with $\beta = \gamma$ which leaves only the rotational component $\sim r$ as in a solid cylinder.

Now let us check what happens when the axisymmetric pinch instability is eliminated by applying a compensating free-space magnetic field with $\beta = \gamma$. In this case, to have any electromagnetic effect on the axisymmetric disturbances considered here, we need to switch on the axial magnetic field by setting $\alpha = 1$. The elimination of the pinch instability turns out to have a surprisingly little effect. Both the critical Reynolds number and frequency, which are shown in Fig. **??**(b) versus the ratio of rotation rates of outer and inner cylinders for $Ha = 10$, look very

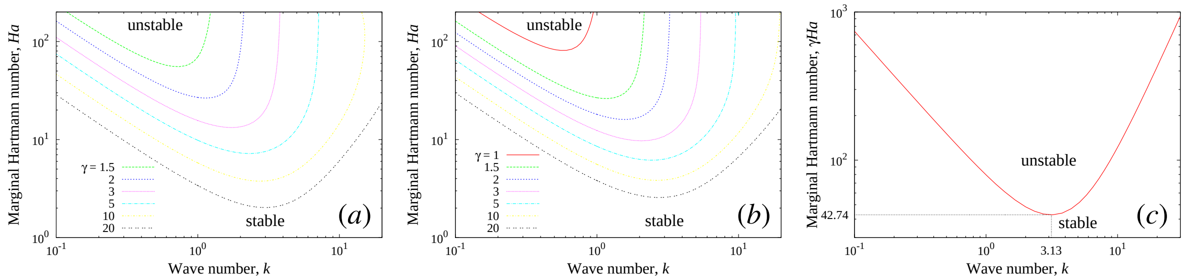

Figure 3: Marginal Hartmann number versus the wave number for purely electromagnetic ($Re = 0$) stationary ($\omega = 0$) instabilities in a rotational magnetic field with $\alpha = 1, \beta = 0$ (a), $\alpha = 1, \beta = \gamma$ (b) and $\alpha = \beta = 0$ (c) at various axial currents defined by γ.

similar to the respective characteristics shown in Fig. **??**(a) for the rotational helical magnetic field with an uncompensated free-space component. As before, the increase of the axial current reduces the critical Reynolds number, which in this case drops to zero at the critical value $\beta = \gamma \approx 2.9$ leading to an unlimited extension of the instability beyond the Rayleigh limit.

Zero marginal Reynolds number means that the instability does not depend on the background flow and is driven entirely by electromagnetic force which is defined by the Hartmann number. The marginal Ha at which neutrally stable purely electromagnetically sustained disturbances of given wave number appear is plotted in Fig. 3 for various axial current parameters γ in helical magnetic field with uncompensated ($\alpha = 1, \beta = 0$) (a) and compensated $\beta = \gamma$ (b) free-space azimuthal components, and in a purely azimuthal field generated only by the axial current in the liquid annulus ($\alpha = \beta = 0$) (c). For the first two helical field configurations, marginal Ha is seen to vary with γ in a similar way. For purely azimuthal field configuration, when the axial field component is absent, it is important to note that the instability is determined by the effective Hartmann number γHa which is independent of γ. As seen in Fig. 3(c), the lowest value $\gamma Ha_c \approx 42.74$ is attained at the critical wave number $k_c \approx 3.13$.

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