DYNAMO EQUATIONS WITH RANDOM COEFFICIENTS

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Abstract : We study the galactic dynamo equations with random alpha-coefficient which takes two different values corresponding to warm gas and hot gas. Probability that alpha-coefficient takes the value which corresponds to the hot gas is p. We obtain a critical value of the probability *p*, for which the dynamo cannot support the magnetic field growth and calculate the growth rates for various statistical moments of magnetic field.

1. Introduction

It is believed that generation of galactic magnetic fields is a result of dynamo based on a joint action of differential rotation and alpha-effect. Intensity of the effects are usually described by dimensionless parameters R_α and R_ω [1].

Alpha-effect depends on the temperature of the interstellar medium. If there are some intensive processes in the galaxy, such as star formation or supernovae explosions which create regions of ionized hydrogen then the turbulent motions including alpha-effect can be changed. In order to include that in galactic dynamo we consider the alpha-effect parameter as a random process such as alpha takes two different values. The first value is connected with warmed atomic hydrogen, and the second one describes the turbulent motions in regions with highly ionized hot gas. The second option occurs with probability *p* which is connected with ratio between hot and warm gas components.

2. The model

We exploit the so-called no-z model which replaces *z*-derivatives (*z*-axis is perpendicular to the disc plane) by some algebraic expressions and obtain the magnetic field component perpendicular to the disc plane from the solenoidality condition [2, 3]:

$$\frac{\partial B_r}{\partial t} = -R_\alpha B_\varphi - \frac{\pi^2}{4} B_r + \lambda^2 \left\{ \frac{\partial}{\partial r}\left(\frac{\partial}{r\partial r}(rB_r) \right) + \frac{1}{r^2}\frac{\partial^2 B_r}{\partial \varphi^2} - \frac{2}{r^2}\frac{\partial B_\varphi}{\partial \varphi} \right\}; \quad (1)$$

$$\frac{\partial B_\varphi}{\partial t} = -R_\omega B_r - \frac{\pi^2}{4} B_\varphi + \lambda^2 \left\{ \frac{\partial}{\partial r}\left(\frac{\partial}{r\partial r}(rB_\varphi) \right) + \frac{1}{r^2}\frac{\partial^2 B_\varphi}{\partial \varphi^2} - \frac{2}{r^2}\frac{\partial B_r}{\partial \varphi} \right\}. \quad (2)$$

where B_r and B_φ are the magnetic field components in the disc plane, R_α is the dimensionless amplitude of alpha-effect, R_ω is the dimensionless amplitude of differential rotation, $\lambda = h/R$ is the disc aspect ratio, where h is the half-thickness of the galaxy disc, R

is its radius. The distances are measured in galactic radii ($0 < r < 1$), and time is measured in $\frac{h^2}{\eta}$ where η is the turbulent diffusivity. A conventional estimate is $R_\alpha \sim 1$, $R_\omega \sim 10$.

We assume that $R_\omega = 10$ and R_α are described by a random law:

$$R_\alpha = \begin{cases} 0.1 \quad with \quad probability \quad p; \\ 1 \quad with \quad probability \quad (1-p). \end{cases} \tag{3}$$

The memory time for R_α is 0.01 or (in some cases) 0.1.

3. Local approach

Here we present the results for the simplest case of an infinitely thin disc and neglect the losses due to diffusion in the disc plane. Then $\lambda = 0$ and the dynamo equations read

$$\frac{dB_r}{dt} = -R_\alpha B_\varphi - \frac{\pi^2}{4} B_r; \tag{4}$$

$$\frac{dB_\varphi}{dt} = -R_\omega B_r - \frac{\pi^2}{4} B_\varphi. \tag{5}$$

We introduce the so-called dynamo number $D = R_\alpha R_\omega$. If generation is weak, i.e. dynamo number is small, magnetic field decays, if however dynamo number exceeds a critical value $D_{cr} \approx 7$, magnetic field grows.
The dynamo equations (3) – (4) can be rewritten in the matrix form:

$$\frac{d}{dt}\left(B_r, B_\varphi\right) = \left(B_r, B_\varphi\right) \begin{pmatrix} -\frac{\pi^2}{4} & -R_\omega \\ -R_\alpha & -\frac{\pi^2}{4} \end{pmatrix}. \tag{6}$$

which can be solved as follows:

$$\vec{B}(n\Delta t) = \vec{B}((n-1)\Delta t)\exp\left(-\frac{\pi^2}{4}\Delta t\right)C_n, \tag{7}$$

where C_n is the transition matrix:

$$C_n = \begin{pmatrix} \cosh\left(\sqrt{R_\alpha R_\omega}\Delta t\right) & -\sqrt{\frac{R_\omega}{R_\alpha}}\sinh\left(\sqrt{R_\alpha R_\omega}\Delta t\right) \\ -\sqrt{\frac{R_\alpha}{R_\omega}}\sinh\left(\sqrt{R_\alpha R_\omega}\Delta t\right) & \cosh\left(\sqrt{R_\alpha R_\omega}\Delta t\right) \end{pmatrix}. \tag{8}$$

and:

$$\vec{B}(n\Delta t) = \vec{B}(0)C_1...C_n\,. \tag{9}$$

Let

$$\tan\theta = \frac{B_\varphi}{B_r}. \tag{10}$$

The angle θ at every time $t_n = n\Delta t$ has some distribution $\pi_n(\theta)$, which can be described by so-called transition probability density:

$$\pi_{n+1}(\theta) = \int_{-\pi/2}^{+\pi/2} p(\theta,\chi)\pi_n(\chi)d\chi. \tag{11}$$

For $n \to \infty$ the distribution function has a limit: $\pi_n(\theta) \to \pi_\infty(\theta)$. Then, the magnetic field grows rate is

$$\gamma = \frac{1}{\Delta t}\left\langle \ln\left\|\vec{w}A_n\right\|\right\rangle - \frac{\pi^2}{4}, \tag{12}$$

where $\vec{w} = (\cos\alpha, \sin\alpha)$ and α has the limit distribution $\pi_\infty(\alpha)$ [4].

4. Results

First of all, we investigate the dynamo equations numerically for various values of p (Fig.1). The magnetic field grows for $p < 0.43$, and decays for higher p.

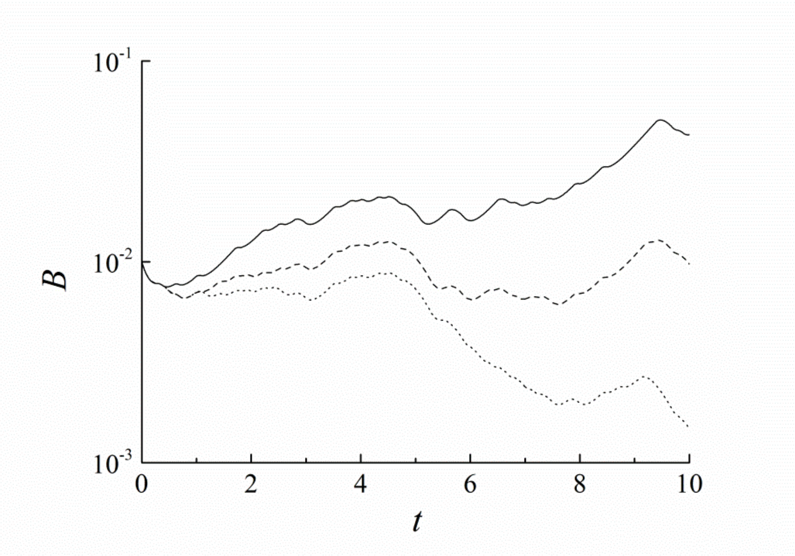

Figure 1: The magnetic field growth for various p. The solid line shows $p = 0.30$, the long-dashed one - $p = 0.40$, the short-dashed one - $p = 0.50$.

The typical growth rates of various statistical momentums of B are given in Table 1. The higher momentums grow faster than lower ones. We found analytically the limiting probability density π (Fig. 2) and calculate the magnetic field growth rate analytically using this density. Numerical and analytical estimates are compared in the Table 1.

We note that the theoretical estimates for the magnetic field growth rate are systematically larger than the numerical ones. Presumably, it means that the theoretical growth rate is determined by very rare random events. Another manifestation of intermittency [5] is the fact that the growth rates of statistical moments grows with the order of the moment.

Table 1: Velocities of different magnetic field momentums growth

	γ_{B}	$\gamma_{\langle \mathrm{B} \rangle}$	$\gamma_{\sqrt{\langle B^2 \rangle}}$	γ_{theor}
$p = 0.30$	0.219	0.224	0.227	0.250
$p = 0.40$	0.040	0.052	0.056	0.065
$p = 0.50$	-0.133	-0.132	-0.128	-0.116

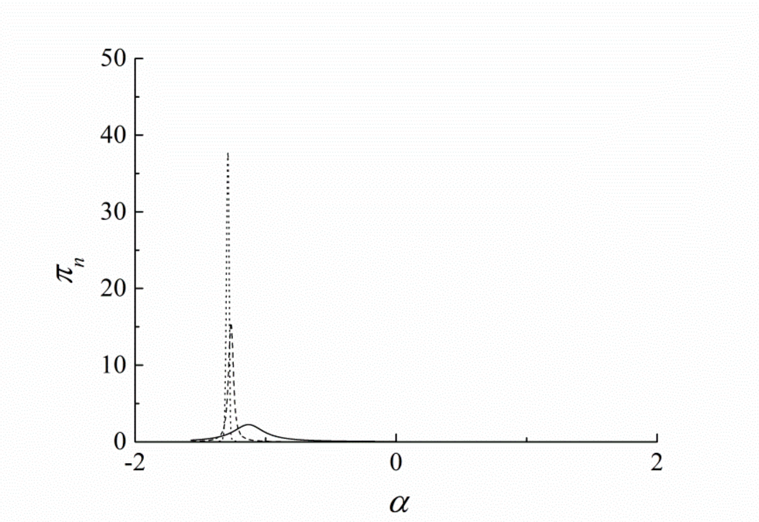

Figure 2: The probability density for different n: the solid line shows $n = 20$, the long-dashed line - $n = 50$, the short-dashed one - $n = 100$.

6. References

[1] Beck, R.; Brandenburg, A.; Moss. D.; Shukurov, A.; Sokoloff, D.: Galactic Magnetism: Recent Developments and Perspectives. Ann. Rev. Astron. Astrophys., 34 (1996), 155 - 206.
[2] Moss, D.: On the generation of bisymmetric magnetic field structures in spiral galaxies by tidal interactions. MNRAS, 275 (1995), 191 - 194.
[3] Phillips, A.: A comparison of the asymptotic and no-z approximations for galactic dynamos. Geophys. Astrophys. Fluid Dynam., 94 (2001), 135 - 151.
[4] Illarionov, E.A.; Sokoloff, D.D.; Tutubalin, V.N.: Stationary distribution of product of matrices with random coefficients. Numerical Methods and Programming, 13 (2012), 218 – 225.
[5] Zel'dovich, Ya.B.; Molchanov, S.A.; Ruzmaikin, A.A.; Sokolov. D.D. Intermittency in random media. Sov. Phys. Usp. 30 (1987), 353–369.