VALIDITY OF QUASI-2D MODELS FOR MAGNETO-CONVECTION

BÜHLER[1] L., MISTRANGELO[1] C., MOLOKOV[2] S.

[1] *Karlsruhe Institute of Technology (KIT)*, Postfach 3640, 76021 Karlsruhe, Germany

[2] *Coventry University,* Priory Street, Coventry CV1 5FB, UK

leo.buehler@kit.edu

Abstract: For applications in nuclear fusion reactors where magnetic fields are very strong, liquid metal flow in the core of ducts can often be regarded as inertialess and practically inviscid, while viscous effects are localized in thin boundary layers. The intense electromagnetic Lorentz forces, resulting from the interaction of induced electric currents and imposed magnetic field, tend to remove flow variations along magnetic field lines and they force the fluid to circulate mainly in planes perpendicular to the field. The established quasi-two dimensional (Q2D) magnetohydrodynamic (MHD) flow can be predicted by means of an approximate model by reducing the basic governing equations to a 2D problem by analytical integration along magnetic field lines. Such models have been applied in the past by numerous authors to investigate duct flow problems and magneto-convection. However, limitations of those Q2D approaches have never been systematically studied.

1. Introduction

Liquid metal flows in strong magnetic fields are dominated by Lorentz forces, and viscous effects are confined to very thin boundary layers. The flow in the inviscid core is highly correlated along magnetic field lines and changes of variables in this direction are often negligible. This fact has been exploited in the past to derive Q2D model equations following the ideas proposed by Sommeria & Moreau (1982) [1]. Q2D models enable an efficient solution of 3D MHD problems, e.g. for shear flow instabilities [2] [3], DNS simulations of Q2D turbulent flows [4], including heat transfer and buoyant flows [5] [6] [7], interpretation of experimental data [8] [9], or simulations for fusion blanket applications [10] etc. It has been shown that results for inertial isothermal flows obtained by the Q2D model can be further improved by a proper modeling of inertia terms, which leads to "barrel" or "cigar" shape flow patterns aligned along the magnetic field [11] [12] instead of pure 2D structures.

The purpose of the present work is showing that Q2D models may have significant deficits for particular classes of buoyant flows, a fact that is not at all obvious from a first point of view. As an example we consider buoyant MHD flows in a horizontal liquid metal layer of height H, length lH and width $2aH$ (see Figure 1). We apply the Q2D model equations and compare results with 3D numerical simulations of full governing equations. Such geometries are typical in horizontal Bridgman crystal growth or for liquid metal blankets of fusion reactors.

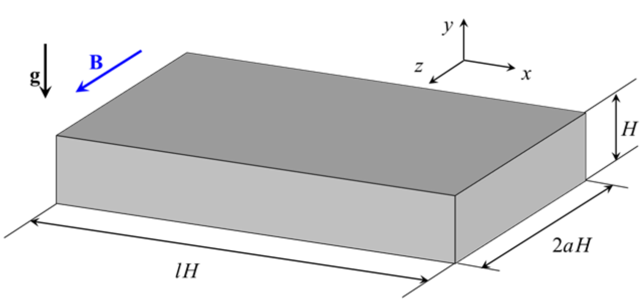

Figure 1 Sketch of geometry and coordinates. The flat cavity, filled with liquid metal, is differentially heated at $x/H=\pm\frac{1}{2}l$, such that a mean axial temperature gradient $G\hat{\mathbf{x}}$ establishes. Top and bottom walls at $y/H=\pm\frac{1}{2}$ have temperature profiles that vary linearly between the values of the differentially heated walls. The other walls are adiabatic. The convective motion is damped by a horizontal magnetic field.

2. Model equations

Buoyant flows of viscous, electrically conducting fluids in a uniform horizontal magnetic field are described by nondimensional equations for balance of energy, momentum and mass, by Ohm's law and by an electric potential equation to ensure charge conservation $\nabla\cdot\mathbf{j} = 0$:

$$Pr\,\mathrm{D}_t T = \nabla^2 T\,, \tag{1}$$

$$D_t\mathbf{u} + \nabla p - \nabla^2\mathbf{u} = GrT\hat{\mathbf{y}} + Ha^2(\mathbf{j}\times\mathbf{B}),\ \nabla\cdot\mathbf{u} = 0, \qquad (2)\ (3)$$

$$\mathbf{j} = -\nabla\phi + \mathbf{u}\times\mathbf{B}\ \text{ and } \nabla^2\phi = \nabla\cdot(\mathbf{u}\times\mathbf{B}) = \mathbf{B}\cdot\boldsymbol{\omega}\,. \qquad (4)\ (5)$$

Here T, $\mathbf{u}$, $\mathbf{B} = \hat{\mathbf{z}}$, $\mathbf{j}$, p and ϕ stand for the temperature difference with respect to a reference value, velocity, magnetic field, current density, pressure and electric potential, scaled by characteristic values ΔT, u_0, B_0, $\sigma u_0 B_0$, $\sigma u_0 B_0^2 H$ and $u_0 B_0 H$, respectively. Dimensionless parameters are the Prandtl number, Grashof number and Hartmann number:

$$Pr = \frac{\nu}{\kappa}, \qquad Gr = \frac{g\beta H^3\Delta T}{\nu^2}, \qquad Ha = B_0 H\sqrt{\frac{\sigma}{\rho\nu}}. \tag{6}$$

Kinematic viscosity ν, thermal diffusivity κ and electric conductivity σ are assumed to be constant, ρ is the density at the reference temperature and β is the volumetric thermal expansion. B_0 is a typical magnitude of the magnetic field, $u_0 = \nu/H$ and ΔT is derived from the mean horizontal temperature gradient $G\hat{\mathbf{x}}$ as $\Delta T = GH$. At walls we have no-slip $\mathbf{u} = 0$. Currents may continue their path inside walls and create there a distribution of wall potential according to the thin-wall condition [13], $\mathbf{j}\cdot\mathbf{n} = c\nabla_w^2\phi_w$, where $c = \sigma_w t_w/(\sigma H)$ stands for the conductance ratio of walls with conductivity σ_w and thickness t_w, ∇_w is the gradient in the plane of the wall and the unit normal $\mathbf{n}$ points into the fluid.

It is well known that for strong magnetic fields, $Ha \gg 1$, the flow takes place preferentially in planes perpendicular to $\mathbf{B}$, i.e. $\mathbf{u} \approx \mathbf{u}_\perp$, and it is described by an equation for the field aligned component ω_z of vorticity $\boldsymbol{\omega} = \nabla\times\mathbf{u}$ that is obtained by taking the curl of (2)

$$(\nabla\times\mathrm{D}_t\mathbf{u}_\perp)_z - \nabla^2\omega_z = Gr\,\partial_x T + Ha^2\partial_z j_z\,. \tag{7}$$

Following the ideas usually referred to as Q2D approach (see [1] and others), the vorticity equation (7) and potential equation (5) are integrated along magnetic field lines (overbar above variables denotes average along field lines);

$$\overline{(\nabla\times\mathrm{D}_t\mathbf{u}_\perp)}_z - \nabla_\perp^2\overline{\omega}_z - \frac{1}{a}\partial_z\omega_z(z=a) = Gr\,\partial_x\overline{T} + Ha^2\frac{1}{a}j_z(z=a), \tag{8}$$

$$\nabla_\perp^2\overline{\phi} + \frac{1}{a}\partial_z\phi(z=a) = \nabla_\perp^2\overline{\phi} - \frac{1}{a}j_z(z=a) = \overline{\omega}_z\,. \tag{9}$$

When Q2D models are applied, it is usually assumed that the potential does not change along magnetic field lines, $\overline{\phi} = \phi(z=a) = \phi_H$. With the thin-wall condition $j_z(z=a) = -c\nabla_\perp^2\phi_H$ [13] and viscous friction $\partial_z\omega_z(z=a) = -Ha\overline{\omega}_z$ applied at the Hartmann wall, j_z and $\overline{\phi} = \phi_H$ can be eliminated from (8) and (9) and the Q2D equation vorticity becomes

$$\overline{(\nabla\times\mathrm{D}_t\mathbf{u}_\perp)}_z - \nabla_\perp^2\overline{\omega}_z = Gr\,\partial_x\overline{T} - \underbrace{\left(\frac{cHa^2}{a+c} + \frac{Ha}{a}\right)}_{1/\tau}\overline{\omega}_z\,. \tag{10}$$

Instead of solving (10) we may solve the following equation, the curl of which yields (10):

$$\overline{\mathrm{D}_t\,\mathbf{u}_\perp} - \nabla_\perp{}^2\overline{\mathbf{u}}_\perp + \nabla_\perp\overline{p} = Gr\overline{T}\hat{\mathbf{y}} - \frac{1}{\tau}\overline{\mathbf{u}}_\perp \text{ with } \nabla\cdot\overline{\mathbf{u}}_\perp = 0. \tag{11}$$

The model derived above is valid only for a *uniform* horizontal temperature gradient as shown in the following. For liquid metals with $Pr \ll 1$ conduction heat governs (1) which supports the ansatz $T=x+Pr\,\theta$, where θ describes deviations from pure heat conduction. Flows with $Gr \gg 1$ and $Ha \gg 1$ are dominated by the right-hand side of (7), through a balance between Lorentz forces and buoyancy, and for $Pr \ll 1$ current density and potential become approximately

$$-\partial_z j_z = \partial_{zz}\phi = Gr/Ha^2\partial_x T \approx Gr/Ha^2. \tag{12}$$

By integration along z the potential ϕ and its mean value $\overline{\phi}$ along z are determined as

$$\phi = \phi_H + \frac{Gr}{2Ha^2}\left(z^2 - a^2\right) \text{ and } \overline{\phi} = \phi_H - \frac{a^2 Gr}{3Ha^2}, \tag{13}$$

where the potential ϕ_H at the Hartmann wall at $z = a$ has been introduced as integration function. Already at leading order the potential ϕ is not at all uniform in the core along field lines. For flows where $\partial_x\overline{T} \neq const$ the last term in equation (13) depends also on (x,y) and finally an additional contribution will appear in (11). Moreover, the electric properties of field aligned walls never enter into the Q2D model, although their conductance may have an essential impact on the global closure of current paths with severe consequences for the flow. This will be shown in the following by some selected examples.

3. Results

Let us first consider flows in a perfectly electrically conducting cavity with $c=\infty$. Results from numerical simulations using Q2D and full 3D equations are compared (the latter ones with up to $8\cdot10^6$ grid points, all layers well resolved, grid-independent results achieved). Figure 2 shows contours of velocity magnitude in the vertical symmetry plane $z=0$ for $a=1$, $Gr=10^8$, $Pr=0.015$, $Ha=1000$. Results deviate by more than one order of magnitude and they are qualitatively quite different. While Q2D solutions show a more or less smooth velocity field, 3D simulations predict a low velocity core and thin boundary layers with very high velocity along the walls at $y=\pm½$ and $x=\pm½l$. However, there is significant disagreement only in layers along those walls. This can be seen by a quantitative comparison of axial velocity profiles as shown in Figure 3. At some distance from the walls Q2D and 3D results in the core agree quite well. Nevertheless, since the layers carry the major mass flux a 3D simulation is mandatory and Q2D results are practically useless as can be seen also by a comparison of temperature profiles in the middle of the cavity (Figure 3). The flow rate in field aligned layers that is missing in the Q2D model can be estimated according to [14] e.g. at the upper wall for a cross-section $x=constant$ as

$$Q_\delta = \int_{-a}^{a}\int_\delta u\,dydz = -\int_{-a}^{a}\int_\delta \partial_y\phi\,dydz = -2a\int_\delta \partial_y\overline{\phi}\,dy = -2a\left(\overline{\phi}_w - \overline{\phi}_\delta\right) = -2a\frac{a^2 Gr}{3Ha^2}. \tag{14}$$

Here $\int_\delta dy$ indicates integration across the layer. For perfectly conducting walls $\overline{\phi}_w = 0$ while the potential $\overline{\phi}_\delta$ at the edge of the layer is given by (13). The vorticity in the core at leading order may be estimated from (10) as $\overline{\omega}_z = \partial_x\overline{v} - \partial_y\overline{u} = \tau\,Gr$, from which the axial core flow rate in the upper half of the cavity results by integration as

$$Q_c = \int_{-a}^{a}\int_0^{½} u\,dydz = -\frac{1}{4}a\frac{Gr}{Ha^2}. \tag{15}$$

This simple estimate shows clearly that the error in not-considering the parallel layers in Q2D models can be significant. Further 3D simulations with perfectly conducting Hartmann walls and insulating field-aligned walls show an additional increase in side layer velocity by another order of magnitude so that a comparison with corresponding Q2D results becomes even worse.

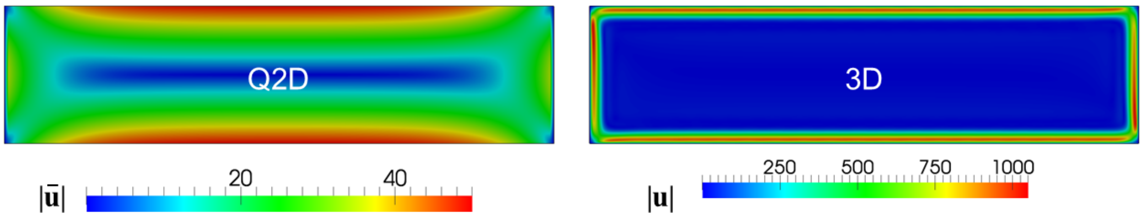

Figure 2 Colored contours of velocity magnitude in the vertical symmetry plane z=0 obtained by Q2D and 3D simulations for a=1, Gr=10^8, Pr=0.015, Ha=1000, c=∞.

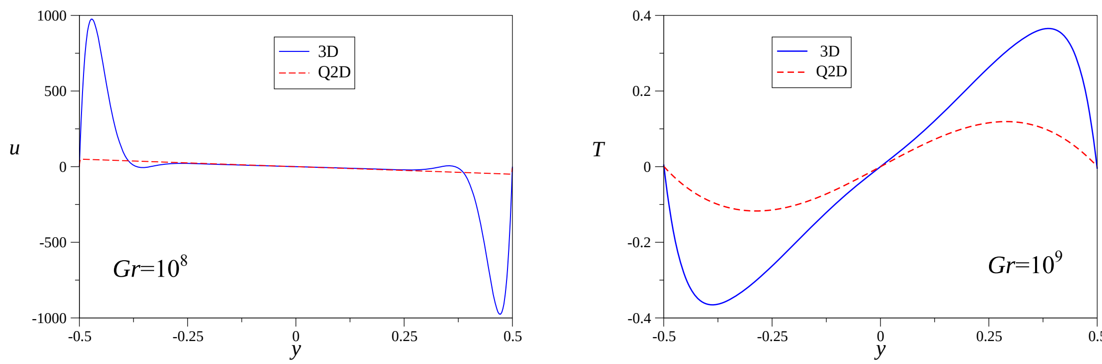

Figure 3 Comparison of axial velocity and temperature along y at (x,z)=(0,0) obtained by Q2D and 3D simulations for a=1, Pr=0.015, Ha=1000, c=∞.

For walls that are poorly conducting or insulating as in [7] the agreement between Q2D and 3D improves because the layer flow rate decreases while simultaneously the core flow rate increases. Results for a=1, Pr=0.015, Gr=10^6, Ha=1000, c=0 are shown in Figure 4. For such parameters the Q2D model is able to predict the velocity magnitude, i.e. results are not as bad as for conducting Hartmann walls. Nevertheless, one can observe still minor differences between the O2D model and 3D simulations.

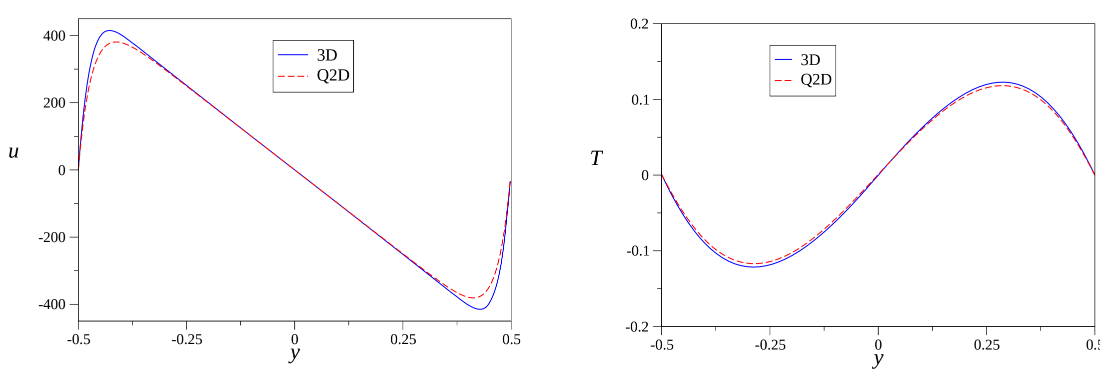

Figure 4 Comparison of axial velocity and temperature along y at (x,z)=(0,0) obtained by Q2D and 3D simulations for a=1, Gr=10^6, Pr=0.015, Ha=1000, c=0.

4. Conclusions

Q2D models have been often applied in the past as efficient tools for numerical simulations of various MHD phenomena for $Ha \gg 1$. It has been shown in the present work that those models may have severe deficits for instance because the electric conductivity of field-aligned walls is not considered. Moreover, for the derivation of Q2D models it is usually assumed that the electric potential is uniform along magnetic field lines, an assumption that is not justified for convection problems. A comparison of results with 3D numerical simulations suggests that for electrically insulating walls Q2D models give reasonable estimates for velocity and heat transfer. For electrically conducting walls, however, Q2D results become useless so that 3D simulations are mandatory.

5. References

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