Turbulent Magnetic Prandtl Number and Spatial Parity Violation

[1]JURČIŠINOVÁ E., [1]JURČIŠIN M., [1,2]REMECKÝ R., and [1]ZALOM P.

[1]Institute of Experimental Physics, Slovak Academy of Sciences, Košice, Slovakia
[2]Bogoliubov Laboratory of Theoretical Physics, JINR, Dubna, Russian Federation

e-mail address for corresponding author: remecky@saske.sk

Abstact: Using the field theoretic RG approach in the two-loop approximation the influence of helicity (spatial parity violation) on the turbulent magnetic Prandtl number is studied in the model of the kinematic MHD turbulence, where the magnetic field behaves as a passive vector quantity advected by the helical turbulent environment given by the stochastic Navier-Stokes equation. It is shown that the presence of helicity decreases the value of the turbulent magnetic Prandtl number and that the two-loop helical contribution to the turbulent magnetic Prandtl number is up to 4.2% of its nonhelical value.

1 Introduction

One of the most important characteristics of the behavior of the magnetic field in a conductive medium is the magnetic Prandtl number, a dimensionless parameter defined as the ratio of the kinematic viscosity to the coefficient of the magnetic diffusivity. The ratio of the turbulent viscosity to the turbulent magnetic diffusivity is the so-called turbulent magnetic Prandtl number $Pr_{m,t}$ [1, 2], which is an analogy to the turbulent Prandtl number of the thermal diffusion [3, 4] and which obtains a universal value in the limit of fully developed turbulence.

Theoretical investigations of the phenomena connected with fully developed turbulence are often based on the renormalization group (RG) methods [4, 5, 6] in the framework of the field theoretic RG technique which is based on the standard formalism of quantum field theory. Due to the fact that the field theoretical models of fully developed turbulent systems belong among the models with strong coupling constants [6, 7], it is also important to calculate the higher-loop corrections at least to estimate the stability and the relevance of the one-loop results with respect to the perturbation corrections.

The complete field theoretic two-loop RG analysis of the genuine MHD turbulence described by the coupled stochastic MHD equations is still missing due to its complexity. Nevertheless, even in this situation the two-loop value of the turbulent magnetic Prandtl number can be studied and estimated by using the so-called kinematic MHD turbulence. In the framework of the kinematic MHD turbulence the Lorentz force term in the corresponding stochastic Navier-Stokes equation is considered negligibly small and the magnetic field behaves as a kind of passively advected vector field (see, e.g., Ref. [8]).

2 Kinematic MHD turbulence

To describe the passive solenoidal magnetic field in a helical turbulent environment we use the model of the kinematic MHD turbulence which is given by the following system of stochastic equations:

$$\partial_t \mathbf{b} = \nu_0 u_0 \Delta \mathbf{b} - (\mathbf{v}\cdot\partial)\mathbf{b} + (\mathbf{b}\cdot\partial)\mathbf{v} + \mathbf{f}^{\mathbf{b}}, \quad (1)$$

$$\partial_t \mathbf{v} = \nu_0 \Delta \mathbf{v} - (\mathbf{v}\cdot\partial)\mathbf{v} - \partial\mathcal{P} + \mathbf{f}^{\mathbf{v}}, \quad (2)$$

where ν_0 is the viscosity coefficient, u_0 is the reciprocal magnetic Prandtl number, $\mathbf{v} \equiv \mathbf{v}(x)$ is an incompressible velocity field, and $\mathcal{P} \equiv \mathcal{P}(x)$ is the pressure. Both $\mathbf{v}$ and $\mathbf{b}$ are divergence-free vector fields, i.e., $\partial \cdot \mathbf{v} = \partial \cdot \mathbf{b} = 0$.

The quantities $\mathbf{f^v}$ and $\mathbf{f^b}$ in Eqs. (1) and (2) are random noises which simulate the corresponding energy pumping into the system to maintain the steady state of the dissipative turbulent environment. In what follows, we suppose that the magnetic energy pumping is given by a transverse Gaussian random noise $\mathbf{f^b} = \mathbf{f^b}(x)$ with zero mean and the correlation function in the form

$$D^b_{ij}(x;0) \equiv \langle f^b_i(x) f^b_j(0)\rangle = \delta(t) C_{ij}(|\mathbf{x}|/L), \tag{3}$$

which represents the source of the fluctuations of the magnetic field. The explicit form of the function C_{ij} in (3) is not essential in what follows, the only condition which must be satisfied is that C_{ij} decreases rapidly for $|\mathbf{x}| \gg L$.

The transverse random force per unit mass $\mathbf{f^v} = \mathbf{f^v}(x)$ simulates the kinetic energy pumping into the system on large scales. It has to be chosen in a form suitable for the description of real infrared energy pumping. In addition, we require the powerlike form of the energy pumping which enables us to apply the RG technique for investigation of the problem [5, 6]. Both conditions are satisfied by the following Gaussian statistics of the random force $\mathbf{f^v}$ with zero mean and pair correlation function:

$$D^v_{ij}(x;0) \equiv \langle f^v_i(x) f^v_j(0)\rangle = \delta(t) \int \frac{d^d\mathbf{k}}{(2\pi)^d} D_0 k^{4-d-2\epsilon} R_{ij}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{4}$$

Geometrical properties of the energy pumping are completely controlled by the form of the transverse projector $R_{ij}(\mathbf{k})$ in (4). In the case of fully symmetric and isotropic energy pumping it is given by the standard transverse projector

$$R_{ij} \equiv P_{ij} = \delta_{ij} - k_i k_j / k^2. \tag{5}$$

On the other hand, in the isotropic but helical case the transverse projector $R_{ij}(\mathbf{k})$ has the following form

$$R_{ij}(\mathbf{k}) = P_{ij}(\mathbf{k}) + H_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j / k^2 + i\rho\varepsilon_{ijl} k_l / k. \tag{6}$$

Here, ε_{ijl} is the Levi-Civita completely antisymmetric tensor of rank 3 and the real parameter, ρ, characterizes the amount of helicity in the system. Due to the requirement of positive definiteness of the correlation function the absolute value of ρ must be in the interval $|\rho| \in [0, 1]$.

3 Field Theoretic Formulation of the Model

The stochastic problem (1)–(4) can be reformulated into a field theoretic model of the double set of fields $\Phi = \{\mathbf{v}, \mathbf{b}, \mathbf{v}', \mathbf{b}'\}$ with the action functional in the following form:

$$S(\Phi) = \frac{1}{2} \int dt_1\, d^d\mathbf{x}_1\, dt_2\, d^d\mathbf{x}_2 \Big[v'_i(x_1) D^v_{ij}(x_1;x_2) v'_j(x_2) + b'_i(x_1) D^b_{ij}(x_1;x_2) b'_j(x_2) \Big]$$
$$+ \int dt\, d^d\mathbf{x} \{ \mathbf{v}'[-\partial_t + \nu_0 \triangle - (\mathbf{v}\cdot\partial)]\mathbf{v} + \mathbf{b}'[-\partial_t \mathbf{b} + \nu_0 u_0 \triangle \mathbf{b} - (\mathbf{v}\cdot\partial)\mathbf{b} + (\mathbf{b}\cdot\partial)\mathbf{v}] \}, \tag{7}$$

where $x_l = (t_l, \mathbf{x}_l)$, $l = 1, 2$, $\mathbf{v}'(x)$ and $\mathbf{b}'(x)$ are auxiliary transverse fields which have the same tensor properties as fields $\mathbf{v}(x)$ and $\mathbf{b}(x)$ and required integrations and summations over dummy indices are assumed.

The field theoretic model (7) corresponds to standard Feynman diagrammatic perturbation theory with a set of bare propagators and vertices. In the present model propagators have the following form

$$\langle b_i' b_j \rangle_0 = \langle b_i b_j' \rangle_0^* = \frac{P_{ij}(\mathbf{k})}{i\omega + \nu_0 u_0 k^2}, \tag{8}$$

$$\langle v_i' v_j \rangle_0 = \langle v_i v_j' \rangle_0^* = \frac{P_{ij}(\mathbf{k})}{i\omega + \nu_0 k^2}, \tag{9}$$

$$\langle b_i b_j \rangle_0 = \frac{C_{ij}(\mathbf{k})}{| - i\omega + \nu_0 u_0 k^2|^2}, \tag{10}$$

$$\langle v_i v_j \rangle_0 = \frac{g_0 \nu_0^3 k^{4-d-2\varepsilon} R_{ij}(\mathbf{k})}{| - i\omega + \nu_0 k^2|^2}. \tag{11}$$

where $C_{ij}(\mathbf{k})$ is the Fourier transform of function $C_{ij}(\mathbf{r}/L)$ in Eq. (3). On the other hand, the triple (interaction) vertices are $b_i'(-v_j\partial_j b_i + b_j \partial_j v_i)$ and $-v_i' v_j \partial_j v_i$.

Let us briefly remind that the formulation of the stochastic problem given by Eqs. (1)–(4) through the field theoretic model with the action functional (7) allows one to use the well-defined field theoretic means, e.g., the RG technique, to analyze the problem. At the same time, the statistical averages of random quantities in the stochastic problem are replaced with the corresponding functional averages with weight $\exp S(\Phi)$ (see, e.g., Ref. [6] for details).

4 The helical magnetic turbulent Prandtl number

The final formulas for the determination of the two-loop inverse turbulent Prandtl number of passively advected scalar field and for the corresponding inverse turbulent magnetic Prandtl number in the framework of the kinematic MHD turbulence have completely the same form [9, 10, 11] and in the language of the kinematic MHD turbulence it reads

$$\begin{aligned} u_{eff} &= u_*^{(1)} \left(1 + \varepsilon \left\{ \frac{1 + u_*^{(1)}}{1 + 2u_*^{(1)}} \left[\lambda - \frac{128(d+2)^2}{3(d-1)^2} \mathcal{B}(u_*^{(1)}) \right] \right.\right. \\ &+ \left.\left. \frac{(2\pi)^d}{S_d} \frac{8(d+2)}{3(d-1)} [a_v - a_b(u_*^{(1)})] \right\} \right), \end{aligned} \tag{12}$$

where λ and $\mathcal{B}(u_*^{(1)})$ are given by the calculation of the corresponding two-loop Feynman diagrams in our helical and isotropic problem and quantities a_v and $a_b(u_*^{(1)})$ are given by the corresponding expansions to the leading order in ε of the scaling functions of the response functions $\langle vv' \rangle$ and $\langle bb' \rangle$ of the velocity field and the magnetic field, respectively (see Ref. [11] for details). Their numerical values for physically the most important three-dimensional case ($d = 3$) are

$$u_*^{(1)} = 1.39297, \tag{13}$$

$$\lambda = -1.0994, \tag{14}$$

$$a_v = -0.047718/(2\pi^2), \tag{15}$$

$$a_b = -0.041389/(2\pi^2), \tag{16}$$

$$\mathcal{B}(u_*^{(1)}, \rho) = -4.4320 \times 10^{-3} - 0.1326 \times 10^{-3} \rho^2, \tag{17}$$

and the two-loop value of the turbulent magnetic Prandtl number obtains the following final explicit dependence on the helicity parameter ρ (for the physical value $\varepsilon = 2$)

$$Pr_{m,t}(\rho) \equiv u_{eff}^{-1} = \frac{1}{1.42046 + 0.06229\rho^2}. \tag{18}$$

In the limit $\rho \to 0$ one comes to the nonhelical value $Pr_{m,t} = 0.7040$. On the other hand, in the fully helical case, i.e., when $|\rho| = 1$, one has $Pr_{m,t} = 0.6744$. In addition, looking at Eq. (18), one can conclude that the turbulent magnetic Prandtl number decreases in a helical turbulent environment, i.e., when the absolute value of the parameter ρ increases. In (Fig. 1) the dependence of the turbulent magnetic Prandtl number on the helicity parameter ρ is compared with the analogous dependence of the turbulent Prandtl number of passively advected scalar quantity.

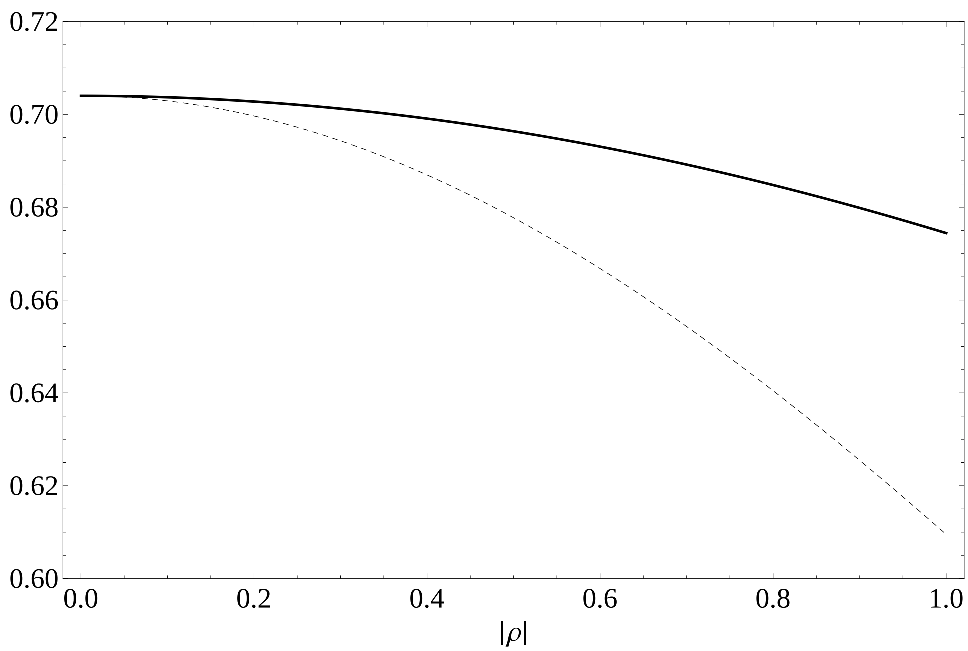

Figure 1: The dependence of the turbulent magnetic Prandtl number $Pr_{m,t}$ (thick line) in the model of the kinematic MHD turbulence and of the turbulent Prandtl number Pr_t (dashed line) of passively advected scalar field on the helicity parameter ρ.

5 Conclusion

In the present paper we have investigated the turbulent magnetic Prandtl number in the framework of the kinematic MHD turbulence under the presence of helicity by using the field theoretic RG technique within the second-order approximation. The explicit dependence ot the $Pr_{m,t}$ on the helicity parameter ρ is found (18).

It was shown that the presence of helicity decreases the value of the turbulent magnetic Prandtl number up to 4.2%, $Pr_{m,t} = 0.6744$ for $|\rho| = 1$, in comparison to its value in the nonhelical system, $Pr_{m,t} = 0.7040$ for $|\rho| = 0$. The fact that the $Pr_{m,t}$ decreases as a function of the absolute value of the helicity parameter also means that the coefficient of turbulent magnetic diffusivity increases as a function of the helicity parameter.

Furthermore we have compared the turbulent magnetic Prandtl number in the helical MHD turbulence to the corresponding turbulent Prandtl number in the model of the passively advected scalar field. In (Fig. 1) one can see that Pr_t and $Pr_{m,t}$ have the same values in the fully symmetric isotropic turbulent environments but are different in the helical systems, i.e., in the systems with spatial parity violation (helicity). It means that the helical turbulent environment distinguishes the internal tensor properties of the advected fields.

The fact that both Prandtl numbers, namely, the turbulent Prandtl number and the turbulent magnetic Prandtl number, decrease as functions of the helicity parameter also means that the corresponding diffusion coefficients increase in helical environments. At the same time, the turbulent diffusion coefficients of scalar fields (temperature field or impurity concentration field) are much more sensitive to the presence of helicity in the system; i.e., their values are essentially more strongly influenced by the helicity than the coefficient of turbulent magnetic diffusivity of the magnetic field in the kinematic MHD turbulence. Therefore, we can conclude that the properties of diffusion processes in the helical turbulent environments can considerably depend on the internal (tensor) properties of the advected fields.

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