The Onset of Rotating Magnetoconvection at Low Ekman

AUJOGUE K.[1], SREENIVASAN B.[1], POTHERAT A.[1]

[1]Applied Mathematic Research Center, Coventry University
aujoguek@uni.coventry.ac.uk

Abstract

A linear stability investigation of rotating magnetoconvection between two differentially heated horizontal planes in a transverse uniform magnetic field at low Ekman number is presented. This study was conducted for Elsasser number Λ (ratio of the Lorentz force to the Coriolis force) from 10^{-3} to 1 and Ekman number E from 10^{-9} to 10^{-2}. Scalings for the critical Rayleigh number and wavenumber at the onset of magnetoconvection were found.

1 Introduction

This work is motivated by applications to the flow in the region of the Earth liquid core magnetoconvection (known as the tangent cylinder). This region seems to play a great role in the structure of the magnetic field and its north pole drift. The main part of the magnetic field is generated by the Earth core liquid flow. To gain better insight on this problem the studies of this flow have relied mostly on DNS [2, 11, 12]. These studies are very expensive and difficult due to the low Ekman and Rayleigh numbers of the order 10^{-14} and 10^{-12} respectively in the Earth core [7]. In order to get closer to more Earthlike regimes, we implemented a linear stability approach. Linear stability is a well known fluids mechanic tool for this kind of problem. Similar problems were envisaged in the past with different basic state on the profil temperature or the magnetic field, and negelecting the inertial and viscous terms in the resolution [3, 4, 10, 6]. These works were conducted for electrically conductive flow rotating in between planes with temperature gradient and horizontal magnetic field. Therefore these configurations are more likely to correspond to the convection outside the tangent cylinder than inside it. We propose here a different kind of basic state closer to tangent cylinder conditions following B. Sreenivasan & C. A. Jones. [11] and Chandrasekhar [5] before them. We found scalings for the different modes of convection. We implemented a new model taking into account the variation of conductivity and thermal diffusivity along the vertical axis. This new linear stablity model was compared with the Chandrasekhar's one [5] for Ekman number $E = 10^{-9}$ and following the phenomenological laws for the Earth core put forward by Pozzo *et al.* [9].

2 Implemented models and results

2.1 Governing equations and geometry

We consider an incompressible fluid (viscosity ν, thermal diffusivity κ, magnetic diffusivity η, density ρ, expansion coefficient α) confined by two differentialy heated infinite horizontal plane boundaries, separated by a distance d. The temperature difference between them is ΔT. The flow rotates at a speed Ω around the vertical axis $\mathbf{z}$ and is subject to a transverse uniform magnetic field $\mathbf{B} = B_0\mathbf{e_z}$. Figure 1 illustrates our geometry.

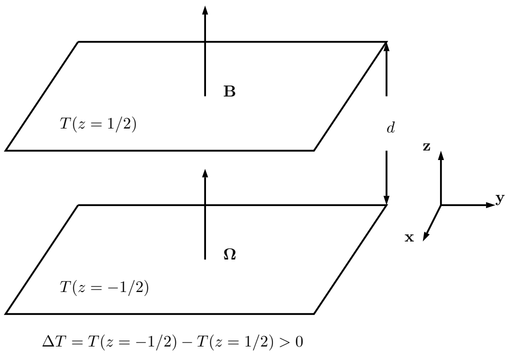

Figure 1: Schematic illustration of the geometry

The flow is governed by the full incompressible MHD equations under the Boussinesq approximation, coupled with energy equation [11, 8]. We normalized lengths by d, the velocity by $\frac{\eta}{d}$, the pressure by $\rho\eta\Omega$, the magnetic field by B_0, the time by $\frac{d^2}{\eta}$, the temperature by ΔT, the rotation speed by Ω. The system becomes controlled by 5 non dimensional parameters: the Ekman number $E = \frac{\nu}{\Omega d^2}$, the Rayleigh number $Ra = \frac{g\alpha\Delta T d}{\eta\Omega}$, the Elsasser number $\Lambda = \frac{B^2}{\mu_0\eta\Omega}$, the Prandtl number $Pr = \frac{\nu}{\kappa}$ and the magnetic Prandtl number $Pm = \frac{\nu}{\eta}$. We applied the two sets of boundary conditions defined by Chandrasekhar [5]: stress free magnetic (SFM) and no-slip magnetic (NSM). For both sets of boundary conditions, the system has a simple solution with $\mathbf{u}_0 = 0$, $\mathbf{B}_0 = 0$, and $T = T_0 + z\Delta T$. We are interested in the linear stability of this basic state. Physical quantities of the problem are decomposed as $g(z) = g_0 + \hat{g}(z)e^{i\mathbf{a}.\mathbf{r}_\perp}$, where $\mathbf{r}_\perp = (x, y)$ and $\mathbf{a}$ is the wave number because of invariance in the $\mathbf{x}$ and $\mathbf{y}$ direction. Following B. Sreenivasan and C. A. Jones [11], we shall only seek the shape of the unstable modes, not their growth rate. The linear stability problem is then solved in the steady state. Furthermore, we shall only consider cases where $Pm = Pr = 1$, under these asumptions, the linearized system is the same described in [11].

$$E(D^2 - a^2)\hat{\omega_z} + 2D\hat{u_z} + \Lambda D\hat{j_z} = 0, \tag{1}$$

$$E(D^2 - a^2)^2\hat{u_z} - 2D\hat{\omega_z} + \Lambda(D^2 - a^2)D\hat{b_z} - 2Ra\hat{T'} = 0, \tag{2}$$

$$(D^2 - a^2)\hat{b_z} + D\hat{u_z} = 0, \tag{3}$$

$$(D^2 - a^2)\hat{j_z} + D\hat{\omega_z} = 0, \tag{4}$$

$$(D^2 - a^2)\hat{T'} + \hat{u_z} = 0, \tag{5}$$

D symbolizes the derivative along $\mathbf{z}$. $\hat{\omega_z} = \boldsymbol{\nabla} \times (\mathbf{u}).\mathbf{e_z}$, $\hat{u_z}$, $\hat{j_z}$, $\hat{b_z}$ and $\hat{T'}$ are the z component complex amplitude of the vorticity, velocity, current, magnetic field, and temperature perturbations respectively and $a =\| \mathbf{a} \|$ is a non dimensional wave number. Following Chandrasekhar [5] and using mass conservation, boundary conditions are expressed as (6, 7).

$$D^2\hat{u_z} = \hat{u_z} = D\hat{\omega_z} = \hat{j_z} = \hat{T'} = 0 \quad \text{for} \quad z = \pm 1/2, \tag{6}$$

$$D\hat{u_z} = \hat{u_z} = \hat{\omega_z} = \hat{j_z} = \hat{T'} = 0 \quad \text{for} \quad z = \pm 1/2\,. \tag{7}$$

The problem becomes a generalized eigenvalue problem of the form $AX = RaBX$. The critical Rayleigh number for the onset of convection Ra_c is found as an eigenvalue of the problem for any given a and minimized over a as in Chandrasekhar [5].

2.2 Resolution

We solved the problem numerically using a spectral collocation method based on Tchebychev Polynomials. In the no-slip case, a boundary layer of the thickness $\delta = 2\sqrt{E\pi}$ develops along the wall, we have ensured that at least 3 collocation points were in it [1]. The validity of the results was guaranteed by convergence tests. We observe a quick convergence to the relative error ϵ over Ra_c at $N = 3000$, the number of collocation points. We made sure that the relative error ϵ was of the same order than the numerical precision.

2.3 Results

In figure 2, we illustrate the typical behaviour of the critical Ra_c with respect to a. For each case, we note three specific values for Ra_c. The first is a minimum occurring at low a, its position and value depends hardly on E but is mostly controlled by Λ. As such, it is referred to as the magnetic mode that we shall denote $(Ra_c^m,\ a_c^m)$ with Ra_c^m the magnetic critical Rayleigh number and a_c^m the magnetic critical wave number. The second is a local minimum for relatively high a, its position and values depend mostly on E. We shall thus refer to it as the viscous mode $(Ra_c^v,\ a_c^v)$ where Ra_c^v is the viscous critical Rayleigh number and a_c^v the viscous critical wave number. The third is a local maximum located between the two previous modes. We call this maximum the intermediate mode $(Ra_c^{int},\ a_c^{int})$. For low E, the value of Ra_c^{int} is several orders of magnitude higher than Ra_c^v and Ra_c^m so a clear separation exists between magnetically controlled modes and those controlled by viscosity.

During the simulations, Λ has been restricted to values below 1 which are relevant to geophysical problems (regime of the Earth core). For higher values of the Elsasser number, B. Sreenivasan and C. A. Jones [11] have shown that the Lorentz force has a stabilizing effect on the flow so that $Ra_c^m(\Lambda)$ increases instead of decreasing as it does for $\Lambda < 1$. For both types of boundary conditions we observed the same results. The system can be characterised by two limits, L_1 when the intermediate and the magnetic mode overlap, $Ra_c^m = Ra_c^{int}$, and L_2 when the most unstable mode switches from magnetic to viscous control, $Ra_c^m = Ra_c^v$. In the limit of $E \to 0$ and $\Lambda \to 0$, we found that L_1 behaves as $\Lambda \propto E^{-1}$ and L_2 as $\Lambda \propto E^{-1/3}$. The behaviour of L_1 is a new result highlighting the transition from a viscous only control system to a magnetic and viscous control system. The scaling for L_2 shows when the magnetic mode becomes more unstable than the viscous mode. This second result respects the scaling found by Chandrasekhar [5] about the viscous mode.

Using the same notation, boundary conditions, and logic, we obtained a linear stability model with variable electrical conductivity and thermal diffusivity along **z**. This model appears as:

$$E(D^2 - a^2)\hat{\omega_z} + 2D\hat{u_z} + \sigma(z)\Lambda_0 D\hat{j_z} = 0, \tag{8}$$

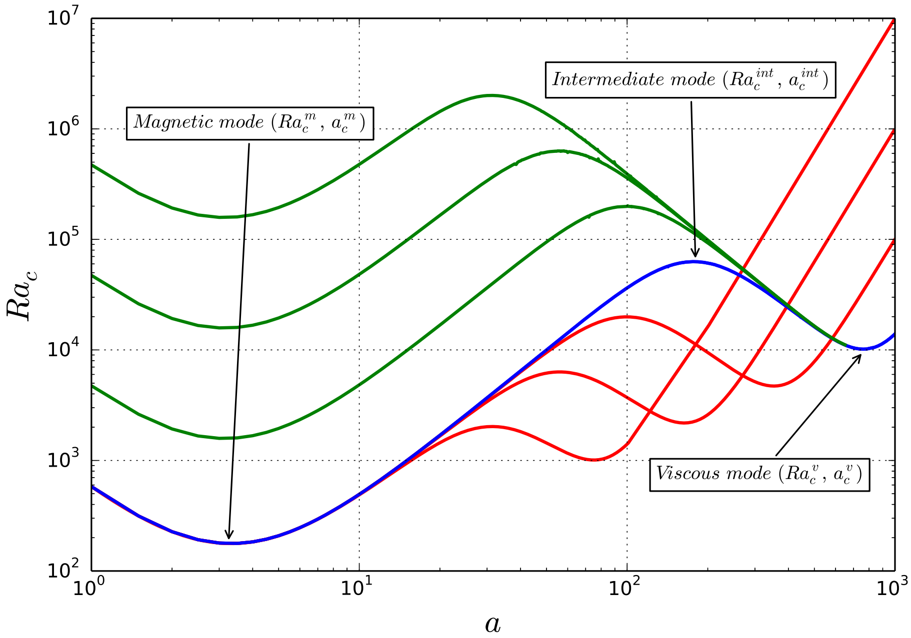

Figure 2: Variation of Ra_c with a. The blue curve's input parameters are $E = 10^{-8}$ and $\Lambda = 1$. The red curves' input parameters are $E = 10^{-8}$ and $\Lambda = 10^{-1}$ to 10^{-3}. The green curves' input parameters are $E = 10^{-5}$ to 10^{-7} at $\Lambda = 1$.

$$E(D^2-a^2)^2\hat{u_z}-2D\hat{\omega_z}+\Lambda(\sigma'(z)-B_0\sigma(z)D)(D^2-a^2)\hat{b_z}-Raa^2(\sigma(z)+T_{eq}(z)\sigma'(z))\hat{T'}=0, \quad (9)$$

$$\frac{1}{\sigma(z)}(D^2-a^2)\hat{b_z}+B_0D\hat{u_z}=0, \quad (10)$$

$$\frac{1}{\sigma(z)}(D^2-a^2)\hat{j_z}+B_0D\hat{\omega_z}+(\frac{1}{\sigma(z)})'D\hat{j_z}=0, \quad (11)$$

$$\sigma(z)\kappa(z)(D^2-a^2)\hat{T'}+\hat{u_z}=0, \quad (12)$$

where $\sigma(z)$, $\kappa(z)$ and $T_{eq}(z)$ are linear functions along $\mathbf{z}$ based on the values given by Pozzo *et al.* [9] at the basic state. We solve this new set of equations as we did for the previous one.

Figure 3 presents the difference in between the new model (with variable κ and ν) and the classic model (uniform basic state). In both cases, the results were obtained with SFM boundary conditions, $E = 10^{-9}$, $\Lambda = 1$, and $a = [1, 2200]$ as input parameters. The new system appears to be generally more unstable. The three modes of interest are pushed to lower values of Ra_c.

3 Discussion

In this paper, the scalings on the different modes for convections were found in the case of the classic model. Asymptotic regimes were reached in the limit of $E \to 0$. For both boundary conditions the behaviour is similar, we only observed a faster convergence to asymptotic regime for SFM in the limit of $E \to 0$. This result is of importance because it suggests that DNS pursued at E of order 10^{-7} with

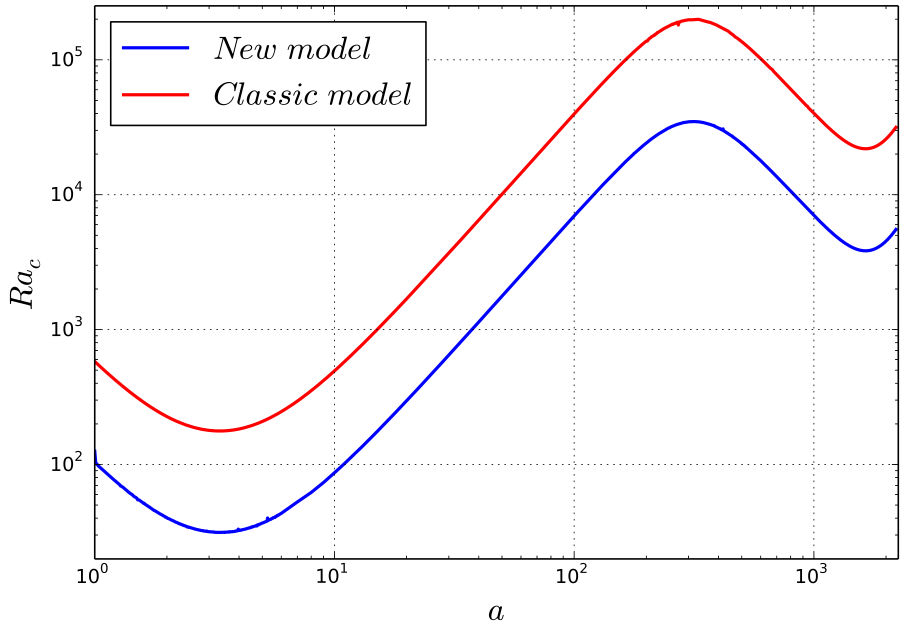

Figure 3: Comparaison in between the two models

SFM provide very accurate insight on the flow inside the tangent cylinder. A new model was implemented showing that variable thermal diffusivity and electrical conductivity generated a more unstable system.

References

[1] D. J. Acheson and R. Hide. Hydromagnetics of rotating fluids. *Reports on Progress in Physics*, 36:159–221, February 1973.

[2] Julien Aubert. Steady zonal flows in spherical shell dynamos. *Journal of Fluid Mechanics*, 542:53–67, 2005.

[3] J Boda. Thermal and magnetically driven instabilities in a non-constantly stratified fluid layer. *Geophysical & Astrophysical Fluid Dynamics*, 44(1-4):77–90, 1988.

[4] Ján Boďa, Gustáv Siráň, and Reviewer P Hejda. Linear stability of a non-constantly stratified horizontal layer penetrated by an azimuthal magnetic field. *Studia geophysica et geodaetica*, 32(3):272–286, 1988.

[5] Subrahmanyan Chandrasekhar. Hydrodynamic and hydromagnetic stability. *International Series of Monographs on Physics, Oxford: Clarendon, 1961*, 1, 1961.

[6] Chen Chuxin and Guo Xiaocheng. Convective instability in a rapidly rotating fluid layer in the presence of a non-uniform magnetic field. *Acta Mechanica Sinica*, 19(6):527–534, 2003.

[7] David Gubbins. The rayleigh number for convection in the earthÕs core. *Physics of the Earth and Planetary Interiors*, 128(1):3–12, 2001.

[8] Peter Olson and Gary A Glatzmaier. Magnetoconvection in a rotating spherical shell: structure of flow in the outer core. *Physics of the Earth and Planetary Interiors*, 92(1):109–118, 1995.

[9] Monica Pozzo, Chris Davies, David Gubbins, and Dario Alfè. Thermal and electrical conductivity of iron at earth/'s core conditions. *Nature*, 485(7398):355–358, 2012.

[10] AM Soward. Thermal and magnetically driven convection in a rapidly rotating fluid layer. *Journal of Fluid Mechanics*, 90(04):669–684, 1979.

[11] Binod Sreenivasan and Chris A Jones. Azimuthal winds, convection and dynamo action in the polar regions of planetary cores. *Geophysical and Astrophysical Fluid Dynamics*, 100(4-5):319–339, 2006.

[12] Binod Sreenivasan and Chris A Jones. The role of inertia in the evolution of spherical dynamos. *Geophysical Journal International*, 164(2):467–476, 2006.