MAGNETO CONVECTIVE INSTABILITIES DRIVEN BY INTERNAL UNIFORM VOLUMETRIC HEATING

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Abstract: A linear stability analysis is performed to investigate the onset of convective motions in a flat cavity filled with liquid metal in which a volumetric heat source is distributed uniformly and a horizontal magnetic field is imposed. A quasi-2D mathematical model is derived by integrating the 3D governing equations along magnetic field direction, which yields a dissipation term in the 2D equations that accounts for 3D viscous effects in thin boundary layers at walls perpendicular to the field. This type of buoyant flow without magnetic field has been investigated by Roberts [1] and the present study extends those results to magnetohydrodynamic conditions. Numerical simulations are performed to support the analytical results and to describe the main convective flow patterns.

1. Introduction

Thermal convective motion produced by uniform internal heat sources in a liquid metal layer, as analyzed in the present study, is a fundamental heat transfer problem of interest for engineering applications such as e.g. nuclear fusion reactors. Here a plasma is confined in a torus by means of a strong magnetic field. Neutron heat is removed by a liquid metal circulating in the so called blanket. Most of the nuclear power is deposited in the liquid metal leading to significantly non-uniform thermal conditions that result in complex convective flow patterns affected by the magnetic field [2] [3].

The problem of natural convection driven by a temperature difference across a fluid layer, the so called Rayleigh-Bénard convection, has been extensively analyzed for applications in crystal growth technology. When the fluid is heated from below it remains motionless until the temperature difference, quantified by the non-dimensional Rayleigh number Ra, exceeds a critical value Ra_{cr} and then thermal convection sets in. Chandrasekhar [4] shows that by applying a magnetic field instabilities occur at higher values of Ra compared to hydrodynamic conditions. At marginal stability convection appears in the form of rolls aligned with the horizontal component of the magnetic field. Analytical and experimental investigations of MHD Bénard-convection can be found in [5]. When a strong magnetic field is applied electromagnetic Lorentz forces elongate vortices along magnetic field lines and force the fluid to move in planes perpendicular to the field, while motion along field lines is damped [6]. This leads to a quasi-two dimensional (Q2D) MHD flow where dissipation losses, due to Joule and viscous effects, are localized in thin Hartmann layers along walls perpendicular to the magnetic field. An explanation of dissipative effects in Hartmann layers is given in [7] [8]. Q2D models reduce the basic governing equations to a 2D problem by analytical integration along magnetic field lines. In the 2D equations 3D MHD effects are modeled by a term that accounts for viscous and Joule's dissipation in Hartmann layers. Those approaches are used to investigate problems related to fusion blankets where intense magnetic fields are present [9].

In the problem studied in this paper convective motions are driven by heat sources distributed in a fluid. The steady laminar hydrodynamic convection in an infinite horizontal fluid layer confined between an isothermal upper plate and a lower one that is thermally insulating has been studied by different authors. This configuration differs from Bénard-convection since temperature boundary conditions are asymmetric and the vertical temperature profile in the motionless state is parabolic rather than linear.

Experimental studies of instabilities in a horizontal fluid layer heated uniformly are described in [10]. Roberts [1] carried out a stability analysis that shows that convective motions occur at $Ra_{cr} \simeq 2772$ in the form of marginally stable rolls. Thirlby [11] performed a numerical analysis and determined the parameters at which polygonal cells and rolls occur in hydrodynamic flows.

The aim of the present study is investigating the influence of a horizontal magnetic field on the onset of instabilities in liquid metal flows with volumetric thermal sources and identifying the main convective patterns. The geometrical configuration chosen for this study is the one used in [1]. Model equations describing the Q2D MHD convective flow are derived (§) and a linear stability analysis is performed (§) to determine the onset of convection depending on intensity of applied heat source and strength of magnetic field. A better understanding of the features of convective flow patterns is obtained by means of numerical simulations.

2. Formulation of the problem and governing equations

Let us consider an electrically conducting liquid metal, filling a horizontal shallow cavity, where a volumetric heat source q is uniformly distributed in the fluid (Figure 1). The top wall at $y = H$ is isothermal, the bottom at $y = 0$ adiabatic, $\partial T/\partial y = 0$, and the Hartmann walls at $z = \pm A$, perpendicular to the magnetic field, are adiabatic, $\partial T/\partial z = 0$.

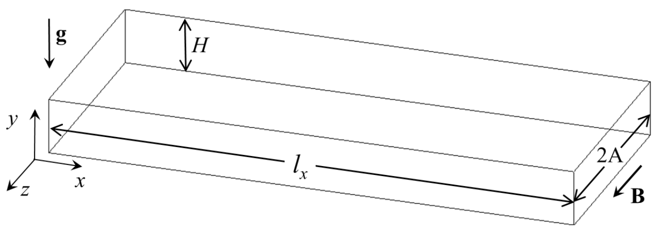

Figure 1. Sketch of geometry and reference system. Walls perpendicular to the magnetic field are thermally insulating, the top wall of the cavity is kept at constant temperature, the bottom is adiabatic. Periodic conditions are assumed in x- direction. A constant magnetic field is applied in horizontal z- direction.

Density changes due to temperature variation are restricted to the buoyancy term, $\rho\beta(T\text{-}T_{ref})\mathbf{g}$, according to the Boussinesq approximation. Here ρ is the density at the reference temperature T_{ref}, β the volumetric thermal expansion coefficient and $\mathbf{g} = -g\mathbf{y}$ the gravitational acceleration. The non-dimensional equations governing the problem account for balance of momentum, conservation of mass and charge and current density is determined by Ohm's law:

$$\frac{1}{Pr}\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right) = -\nabla p + \nabla^2\mathbf{v} + RaT\hat{\mathbf{y}} + Ha^2(\mathbf{j}\times \qquad (1)$$

$$\nabla\cdot\mathbf{v} = 0 \qquad \nabla\cdot\mathbf{j} = 0 \qquad \mathbf{j} = -\nabla\phi + \mathbf{v}\times\mathbf{B}. \qquad (2)\ (3)\ (4)$$

The temperature distribution is given by the energy balance equation

$$\frac{\partial T}{\partial t} + (\mathbf{v}\cdot\nabla)T = \nabla^2 T + 1 \qquad (5)$$

The dimensional volumetric heat source q has been scaled by $\lambda\Delta T/H^2$ and normalized to 1 by defining the characteristic temperature difference as $\Delta T = qH^2/\lambda$. The dimensionless variables $\mathbf{v}$, t, $\mathbf{j}$, ϕ and $\mathbf{B}$ are obtained by scaling velocity, time, electric current density, electric potential and magnetic field by the reference quantities $v_0 = \alpha/H$, H^2/α, $\sigma v_0 B_0$, $v_0 B_0 H$ and B_0, respectively. The typical length scale H is the distance between horizontal walls. Thermal diffusivity $\alpha = \lambda/(\rho c_p)$, thermal conductivity λ, specific heat c_p, kinematic viscosity ν and electric conductivity σ are assumed to be constant in the temperature range considered. The non-dimensional temperature T is given by $(T^* - T_{ref})/\Delta T$, where T^* is the local dimensional temperature. The dimensionless parameters that control the flow are Prandtl number Pr, Rayleigh number Ra and Hartmann number Ha:

$$Pr = \frac{\nu}{\alpha}, \qquad Ra = \frac{g\beta q H^5 \rho c_p}{\nu\lambda^2}, \qquad Ha = B_0 H\sqrt{\frac{\sigma}{\rho\nu}}. \tag{6}$$

The Prandtl number represents the rate of momentum diffusion to the one of heat diffusion. The Rayleigh number describes the intensity of the applied heating. The Hartmann number gives a non-dimensional measure for the strength of the magnetic field. In order to quantify the magnitude of convection we introduce a quantity defined as the ratio between mean temperature differences across the fluid layer without motion and with convection [1] [11]:

$$M = \frac{\int_V T_{cond}\, dV}{\int_V T_{conv}\, dV} = \frac{1}{3\overline{\overline{T}}}. \tag{7}$$

3. 2D model equations

The procedure followed for the analysis is analogous to the one used in [5] [7] [8]. Starting from the 3D equations (§) an equation for vorticity $\boldsymbol{\omega} = \nabla\times\mathbf{v}$ is obtained. For the given boundary conditions and an applied magnetic field in z-direction the flow is characterized by a Q2D velocity $\mathbf{v} = (u,v,0)$ and vorticity $\boldsymbol{\omega} = (0,0,\omega)$, where u, v, ω may depend on (x,y,z):

$$\frac{1}{Pr}\left(\partial_t\omega + u\partial_x\omega + v\partial_y\omega\right) = \nabla^2\omega + Ha^2\partial_z j_z + Ra\partial_x T. \tag{8}$$

In a quasi 2D flow velocity and vorticity can be expressed by a separation ansatz, e.g. $u = \hat{u}(t,x,y)f(z)$. By integrating the vorticity equation along field lines, with no slip at Hartmann walls, $f(z = \pm a) = 0$, and thin wall condition [12] we obtain

$$\frac{1}{Pr}\left(\partial_t\hat{\omega} + \hat{u}\partial_x\hat{\omega} + \hat{v}\partial_y\hat{\omega}\right) = \nabla_{xy}^{\ 2}\hat{\omega} - \frac{1}{\tau}\hat{\omega} + Ra\partial_x T \quad \text{with} \quad \frac{1}{\tau} = \frac{Ha}{a} + \frac{cHa^2}{a+c}, \tag{9}$$

where $c = \sigma_w t_w/(\sigma H)$ is the conductance parameter. In (9) terms on the left hand side represent convective transport of vorticity and its time variation, on the right hand side there are two dissipation terms. The first one describes viscous losses due to gradients of vorticity in a plane perpendicular to **B**. The term $-\hat{\omega}/\tau$ represents viscous and Joule dissipation in the Hartmann layer and in the wall and τ is related to a typical decay time of vorticity [7]. For electrically insulating Hartmann walls ($c = 0$) $1/\tau \to Ha/a$ and for perfectly conducting walls ($c = \infty$) $1/\tau \to Ha^2$, namely in ducts with highly electrically conducting walls a rapid damping occurs.

4. Linear stability analysis

The basic steady state in the problem studied is motionless and with a parabolic temperature distribution along the vertical coordinate y. When the internal heat source, i.e. Ra, is large enough the base state loses its stability due to increased buoyancy forces that are not balanced anymore by viscous effects and thermal conductivity. Convective motions occur whose intensity depends on the internal heat source (Ra) and the magnetic field (Ha). A linear stability analysis is performed that consists in following the evolution of small perturbations applied to the equilibrium state by linearizing equation (9). The stability is determined by solving the resulting eigenvalue problem. In order to derive disturbance equations, temperature, velocity and vorticity are decomposed as the sum of a basic state denoted by the subscript 0 and a perturbation indicated by prime and multiplied by a small parameter ε, e.g. $T = T_0 + \varepsilon T'$. Those expressions are introduced in (5) and (9) and terms of $O(\varepsilon^2)$ are neglected in the small perturbation limit. We expand perturbations in normal modes as e.g. $T' = i\Theta(y)e^{st+ikx}$ where k is a real horizontal wavenumber, s the temporal rate of growth of the perturbation. In order to satisfy mass conservation (2) we introduce a streamfunction $\psi'(x,y)$, such that $\mathbf{u}' = \nabla\times(\psi'\cdot\hat{\mathbf{z}})$ and $\omega' = -\nabla^2\psi'$. After some mathematical work and at the stability limit the equations describing the stability of the problem become:

$$\left[\left(D^2-k^2\right)-\frac{1}{\tau}\right]\Omega-kRa\Theta=0,\qquad \left(D^2-k^2\right)\Theta-ky\Psi=0,\qquad \left(D^2-k^2\right)\Psi+\Omega=0, \tag{10}$$

where $D^2 = \partial^2/\partial y^2$ and Ω, Θ, Ψ are the amplitude functions of vorticity, temperature and streamfunction perturbations, respectively. A numerical procedure has been implemented in Matlab where finite difference techniques are used for the solution of the eigenvalue problem.

5. Results

For $1/\tau \rightarrow 0$ the problem is equivalent to the hydrodynamic flow (Ha = 0) considered in [1], for which Ra_{cr} = 2772 and k_{cr} = 2.63 are predicted. This case is first investigated to validate the used numerical model. In a second step a uniform horizontal magnetic field is imposed and its influence on the stability of the considered magneto-convective flow is studied. Numerical simulations are also performed both to confirm the linear stability analysis and to complement the results by means of 3D and Q2D nonlinear solutions for $Ra > Ra_{cr}$.

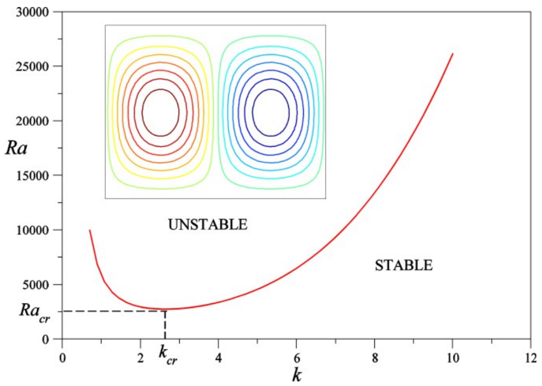

Figure 2. Neutral stability curve for the hydrodynamic flow (Ha = 0) showing marginal Rayleigh number as a function of the wavenumber.

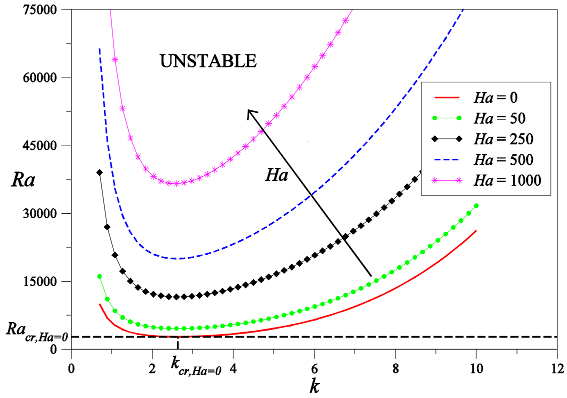

Figure 3. Neutral stability curves for the case of electrically insulating walls, c = 0, and various Hartmann numbers Ha.

Figure 2 shows the neutral stability curve for Ha = 0. As the control parameter Ra increases, the base state first becomes linearly unstable at $Ra = Ra_{cr}$ with respect to perturbations with horizontal wavenumber $k = k_{cr}$. Results agree very well with those presented in [1]. In the region below the curve perturbations of any wavelength decrease, namely the flow is stable, no convective motion occurs and the temperature has a parabolic profile between the plates. Above the curve the flow is unstable and convection occurs with a certain range of unstable wave numbers. In Figure 2 contours of the streamfunction are depicted showing two counter rotating convective cells that appear at the instability threshold.

Let us consider now magneto-convective flows in an electrically insulating cavity (c = 0) with A/H = 2. We analyze the influence of the magnetic field strength (Ha). In Figure 3 neutral stability curves are depicted for various Ha. The curve for the hydrodynamic case (Ha = 0) is shown for comparison. It can be seen that, as expected [4], the magnetic field stabilizes the flow, i.e. by increasing Ha the onset of convection occurs at higher values of Ra.

We fix now the Hartmann number Ha = 200 ($1/\tau$ = 100). The axial length l_x (see Figure 1) is chosen such that 8 convective cells fit in the computational domain at Ra_{cr}. Numerical simulations are performed by using the Q2D model described in §3 that have been implemented in the finite volume code OpenFOAM.

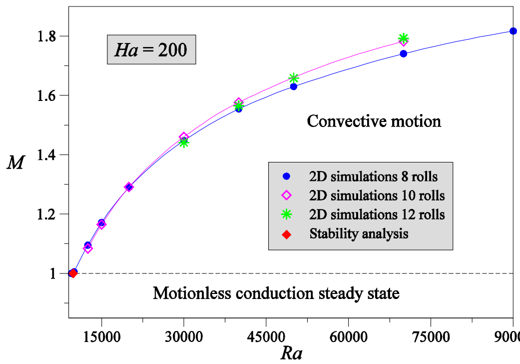

Figure 4. Factor M that quantifies the strength of the convective motion as a function of the Rayleigh number.

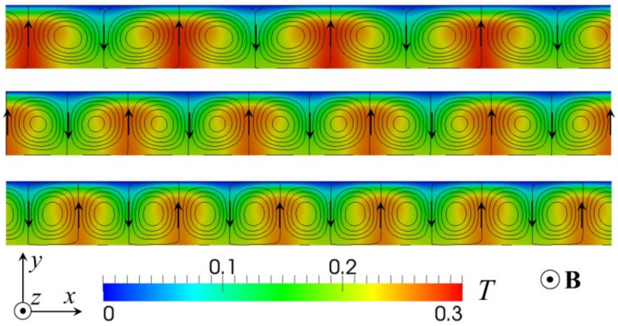

Figure 5. Contours of scaled temperature and electric potential isolines (non-dimensional distance 0.9) for Q2D MHD flow at Ha = 200 and Ra = 70000.

In order to quantify the intensity of the convective motion the quantity M (7) is calculated and plotted as a function of Ra in Figure 4. The numerically predicted critical Rayleigh number, $Ra_{cr} = Ra(M = 1)$, agrees very well with the one obtained by the stability analysis, Ra_{cr} = 9821.3. By increasing the Rayleigh number M becomes larger, i.e. the convective heat transfer intensifies. For sufficiently large Ra various solutions coexist characterized by 8, 10 and 12 rolls. An example is shown in Figure 5 for the flow at Ra = 70000. Here contours of temperature T and isolines of electric potential, which serve as approximate streamlines, are depicted on the cross-section of the cavity.

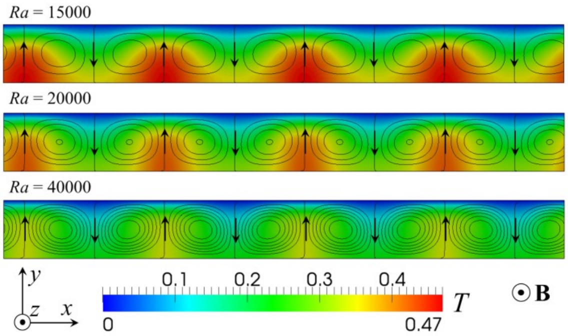

Figure 6. Contours of scaled temperature and electric potential isolines (non-dimensional distance 0.56) for Q2D MHD flows at Ha = 200 and different Ra.

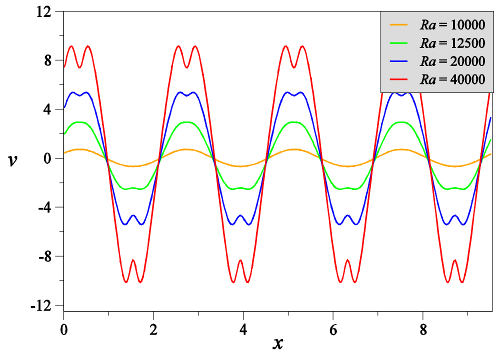

Figure 7. Vertical velocity as a function of the coordinate x for various Ra.

In Figure 6 contours of temperature and isolines of potential are compared for three Ra. In the selected scale for the temperature (see §2) ΔT is proportional to the volumetric heat source and therefore it increases with Ra and the normalized temperature reduces. When the heating is stronger the motion intensifies as indicated by a larger number of potential isolines. In Figure 7 the vertical component of the velocity is plotted for various Ra along the axial coordinate on a line at y = 0.5. When approaching the stability limit (Ra_{cr} = 9821.3) the velocity profile resembles a harmonic function. By rising Ra additional modes appear due to nonlinear interactions leading to more complex velocity profiles.

6. Conclusions

The influence of a magnetic field on the stability of convective flows caused by a uniform volumetric heat source distributed in a liquid metal layer has been studied. A linear stability analysis is performed based on a Q2D model and the critical Rayleigh number for the onset of convection has been calculated for increasing Ha. The occurrence of convection is delayed when Ha becomes larger. For supercritical conditions solutions with different wavenumbers coexist. The velocity distribution shows that when convective motion sets in only harmonic functions contribute and by increasing *Ra* higher modes appear due to nonlinear interactions.

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