Theme 9: SPACE

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Mathematical modelling and numerical simulation in thermoacoustics

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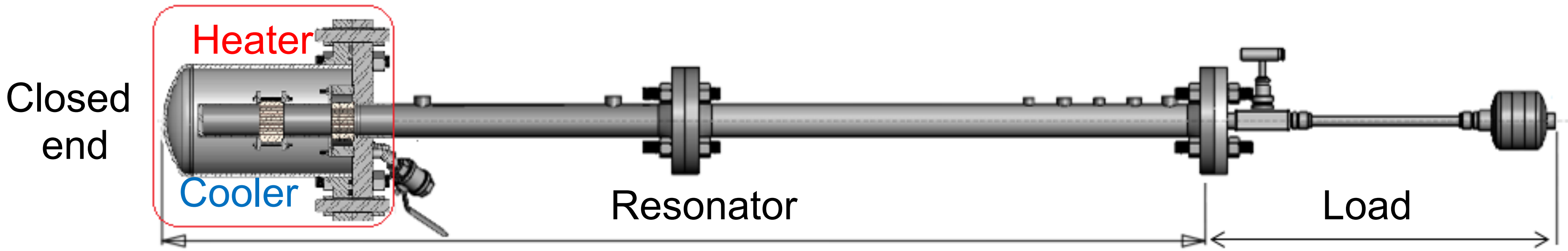
Outline

- **Introduction**
- **Models and simulations for an entire thermoacoustic device:**
  - **Linear acoustic theory and 1D numerical simulations**
  - **1D time dependent models**
  - **Full Navier Stokes equations and associated 2D/3D numerical simulations**
  - **Hybrid model: Low Mach number approximation and 2D numerical simulations**
- **Models and simulations for specific components of thermoacoustic devices**
- **Conclusion**

Introduction

Example of an entire thermoacoustic device:
standing wave thermoacoustic engine / variable load
(demonstrator by **Hekyom** society - France),



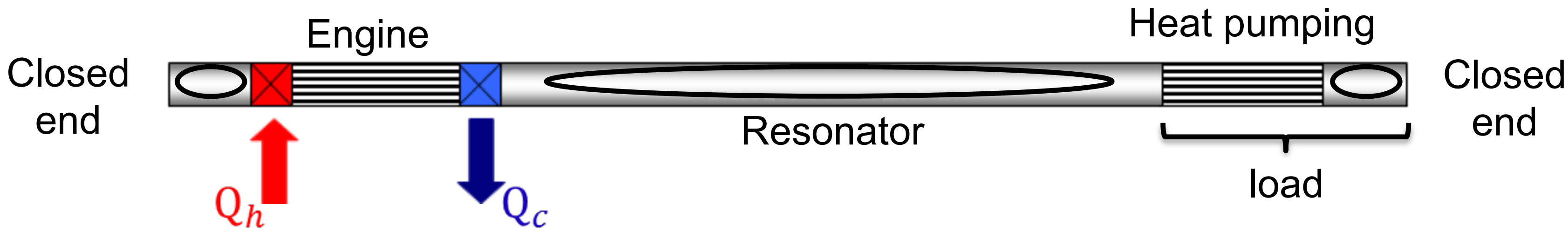
Resonator filled with gas (He, N_2), at rest, under pressure

Stack/porous media + 2 heat exchangers $\Rightarrow$ ΔT

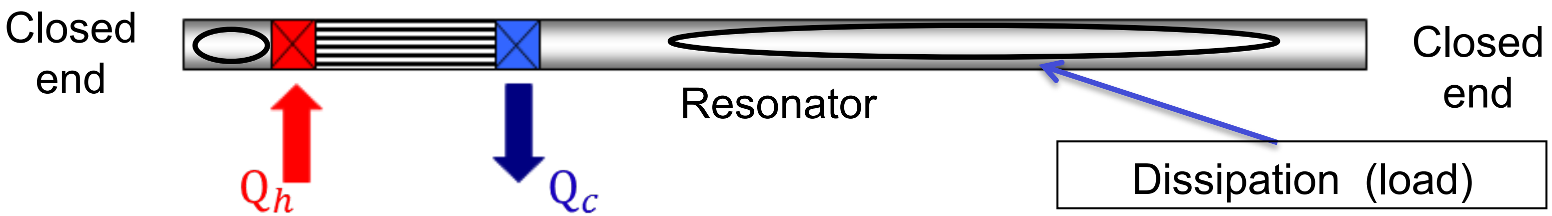
+ initial perturbation $\Rightarrow$ wave amplification

Heat $\Rightarrow$ acoustic power

- Exp. LIMSI : resonator closed at both ends with a load



- Exp. A. Atchley et al, 1992-95 : resonator closed at both ends.



Exp.	L_{res} (m)	L_{st} (cm)	H (mm)	$t_{acoustic}$ (ms)	$t_{thermal}$ (min)	P_m (bar)	Gaz
LIMSI	7.57	15	0.97	7.5	21	10	He
Atchley	1.11	3.5	1.06	1	0.5	4.4	He

Relative position of the stack l_L / L_{res} LIMSI $l_L / L_{res} = 0.089$ Atchley $l_L / L_{res} = 0.13$

- **Problem:**

How to describe:

- gas oscillations (self sustained acoustic oscillations at the frequency of the most unstable mode)
- heat conduction in the gas and the solid parts
- oscillating flow through variable geometry (small passages and wide tubes)

Thermoacoustics = compressible fluid dynamics (acoustics +heat transfer +hydrodynamics)

- **Difficulties: different space scales, times scales, order of magnitude?**

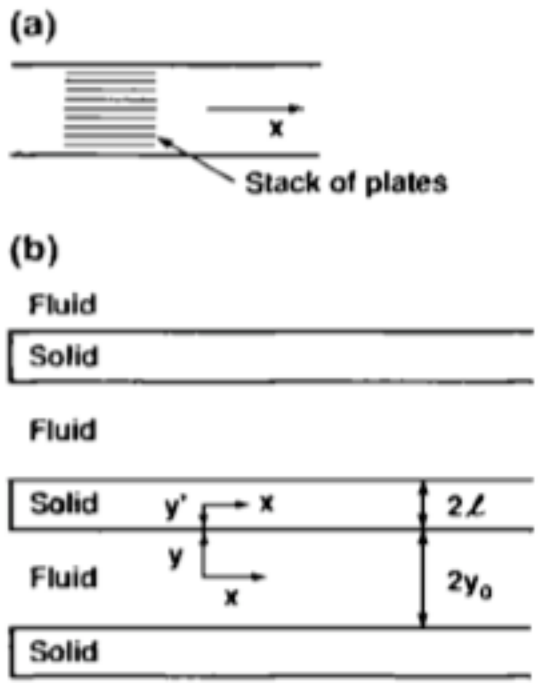
Therefore, description of fine physical phenomena and functioning of entire devices requires:

- very fine spatial meshes over very large domains
- very fine time steps over very long times

$\longrightarrow$ necessity of developing simplified models

Models and simulations for an entire thermoacoustic device

Rott's linear theory (1969-1980)



- Compressible 2D Navier-Stokes Eqns
- Hypotheses:
  - small Mach number (linear acoustics)
  - $\partial / \partial x << \partial / \partial y$ (boundary layer approx)
  - mono-frequency plane wave (angular frequency ω)
  - mean temperature gradient (in the stack)
  - integration along y

- 3 first order ODEs in the frequency domain
  - relating fluctuations of pressure, velocity, temperature, ω et total energy
  - using initial conditions (at rest) of pressure, temperature and density, physical properties of the gas and solid (c_p, ν, k)
  - taking into account each element geometry (stack, heat exchanger, tube portion) through its porosity and form factors in each element.

- **Reference scales:**

$$L_{ref} = L_{res},\ p_{ref} = p_0,\ T_{ref} = T_0,\ U_{ref} = U_0,\ t_{ref} = L_{ref}/c_0,\ c_0 = \sqrt{\left(\partial p/\partial \rho\right)_{s,0}}$$

$$\rho_{ref} = \rho_0,\ \mu_{ref} = \mu_0,\ k_{ref} = k_0,\ c_{p,ref} = c_{p,0}$$

- **Non dimensional parameters:**

$$M = \frac{U_0}{c_0},\ Re = \frac{\rho_0 U_0 L_{res}}{\mu_0},\ Pe = \frac{\rho_0 c_{p0} U_0 L_{res}}{k_0} = Re \cdot Pr,\ \varepsilon = \frac{\delta_\nu}{L_{res}},\ \delta_\nu = \sqrt{\frac{2\nu}{\omega}},\ \Gamma = \frac{\gamma - 1}{\gamma}$$

- **Compressible dimensionless Navier-Stokes equations**

$$\frac{\partial \rho}{\partial t} + M\frac{\partial}{\partial x}(\rho u) + M\frac{\partial}{\partial y}(\rho v) = 0$$

$$\rho\frac{\partial u}{\partial t} + M\rho u\frac{\partial u}{\partial x} + M\rho v\frac{\partial u}{\partial y} = -\frac{1}{\gamma M}\frac{\partial p}{\partial x} + \mu\frac{M}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \frac{1}{\varepsilon^2}\frac{\partial^2 u}{\partial y^2}\right) + \mu\left(\frac{\xi}{\mu} + \frac{1}{3}\right)\frac{M}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y}\right)$$

$$\rho\frac{\partial T}{\partial t} + M\rho u\frac{\partial T}{\partial x} + M\rho v\frac{\partial T}{\partial y} = \Gamma\frac{\partial p}{\partial t} + M\Gamma u\frac{\partial p}{\partial x} + M\Gamma u\frac{\partial p}{\partial y} + M\frac{1}{\varepsilon^2 Pe}\frac{\partial^2 T}{\partial y^2} + \text{quadratic velocity terms}$$

$$p = \rho T$$

Small parameters: $M = \dfrac{U_0}{c_0} << 1, \varepsilon = \dfrac{\delta_\nu}{L_{res}} << 1$

Within the boundary layer: $\varepsilon^2 = O(M), \quad 1 << M\,Re$

Asymptotic developments in powers of *M*

$$< \bullet > = < \bullet >^{(0)} + M < \bullet >^{(1)} + o(M)$$

$$\rho^{(0)} = \rho^{(0)}(x), p^{(0)} = const = \rho^{(0)} T^{(0)}$$

Continuity $$\frac{\partial \rho^{(1)}}{\partial t} + \frac{\partial}{\partial x}(\rho^{(0)} u^{(0)}) + \frac{\partial}{\partial y}(\rho^{(0)} v^{(0)}) = 0$$

Momentum $$\rho^{(0)} \frac{\partial u^{(0)}}{\partial t} = -\frac{1}{\gamma}\frac{\partial p^{(1)}}{\partial x} + \frac{M}{\varepsilon^2 Re}\frac{\partial^2 u^{(0)}}{\partial y^2}$$

Energy $$\rho^{(0)} \frac{\partial T^{(1)}}{\partial t} + \rho^{(0)} u^{(0)} \frac{\partial T^{(0)}}{\partial x} = \Gamma \frac{\partial p^{(1)}}{\partial t} + \frac{1}{\varepsilon^2 Pe}\frac{\partial^2 T^{(0)}}{\partial y^2}$$

State law $$p^{(1)} = \rho^{(0)} T^{(1)} + \rho^{(1)} T^{(0)}$$

Seek solution in harmonic form $(\xi_j(x,y,t) = \Re e\left\{\xi_c^{(j)}(x,y)e^{i\omega t}\right\}),\ \ j=0,1$

Continuity $$i\omega\rho_c^{(1)} + \frac{\partial}{\partial x}(\rho^{(0)}u_c^{(0)}) + \rho^{(0)}\frac{\partial v_c^{(0)}}{\partial y} = 0$$

Momentum $$\rho^{(0)}i\omega u_c^{(0)} = -\frac{1}{\gamma}\frac{\partial p_c^{(1)}}{\partial x} + \frac{\mu M}{\varepsilon^2 \mathrm{Re}}\frac{\partial^2 u_c^{(0)}}{\partial y^2}$$

Energy $$\rho^{(0)}c_p(i\omega T_c^{(1)} + u_c^{(0)}\frac{\partial T^{(0)}}{\partial x}) - i\omega\beta p_c^{(1)} = \frac{k}{\varepsilon^2 \mathrm{Pe}}\frac{\partial^2 T_c^{(1)}}{\partial y^2}$$

Back to dimensional variables

- solve the the y dependence of $u_c^{(1)}, T_c^{(1)}$

$$u_c^{(0)} = \frac{i}{\omega\rho^{(0)}}\left[1-h_\nu(y)\right]\frac{\partial p^{(1)}}{\partial x}$$ and similar for temperature

- integration along y

$$U_1 = \int U_0 u_c^{(0)}\,dy,\quad p_1 = p_0 p_c^{(1)},\ T_j = \int T_0 T_c^{(j)}\,dy$$

$\Longrightarrow$ Rott’s equations (Rott 1980, Swift1998)

Rott's equations (Rott 1980, Swift1998)

$$\frac{dp_1}{dx} = -\frac{1}{A}\frac{i\omega\rho_0}{1-f_\nu}U_1$$

$$\frac{dU_1}{dx} = -\frac{i\omega A}{\gamma p_0}[1+(\gamma-1)f_\kappa]p_1 + \frac{f_\kappa - f_\nu}{(1-f_\nu)(1-\mathrm{Pr})}\frac{1}{T_0}\frac{dT_0}{dx}U_1$$

$$\dot{H}_2 = \frac{1}{2}\mathrm{Re}\left[p_1 U_1{}^*\left(1-\frac{f_\kappa - f_\nu{}^*}{(1-f_\nu{}^*)(1+\mathrm{Pr})}\right)\right] + \frac{\rho_0\left|U_1\right|^2}{2A\omega(1-\mathrm{Pr}^2)\left|1-f_\nu\right|^2}\mathrm{Im}\left(f_\kappa + \mathrm{Pr}\, f_\nu{}^*\right)\frac{dT_0}{dx}$$
$$-(A + A_{solid}k_{solid})\frac{dT_0}{dx}$$

f_ν, f_κ = thermoviscous functions (stack geometry dependent)

$$H_2(x) = \frac{1}{2}\rho_m \Re e\left[h_1 U_1^*\right] - (Ak + A_{solid}k_{solid})\frac{dT_m}{dx} \qquad \text{(total power)}$$

$$E_2(x) = \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} \Re e\left\{p_1(x)e^{i\omega t}\right\}\Re e\left\{U_1 e^{i\omega t}\right\}dt = \frac{1}{2}\Re e\left[p_1 U_1^*\right] \qquad \text{(acoustic power)}$$

Thermoviscous functions for different geometries:

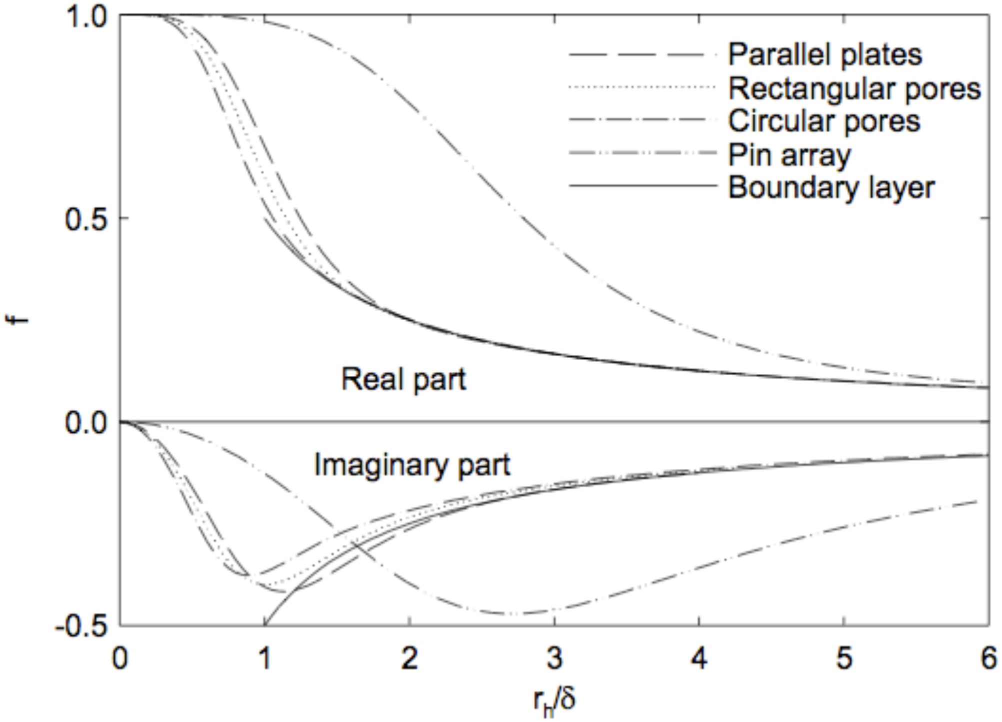


Figure 4.14: Spatial-average function f for several geometries. The rectangle here has 6:1 aspect ratio, and the pin array has $r_o/r_i = 6$. The boundary-layer limit is approached at large r_h in all geometries. Using r_h/δ_ν on the horizontal axis yields f_ν on the vertical axis, and using r_h/δ_κ yields f_κ.

G.W. Swift, « Thermoacoustics: a unifying perspective for some engines and refrigerators, 2002

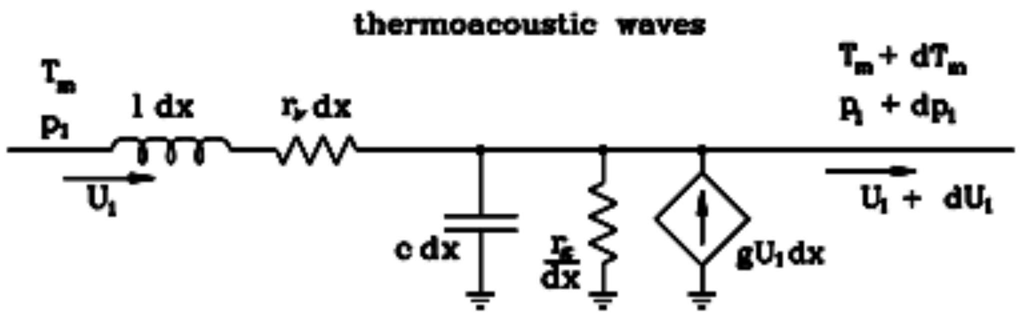
Electro-acoustic analogy

Acoustic networks	**Electric networks**
Pressure	Voltage
Volume flow rate	Current
Compliance	Capacitance
Inertance	Inductance
Resistance	Resistance

$$\frac{dp_1}{dx} = -\frac{i\omega\rho_0}{1-f_\nu}\frac{U_1}{A} \;\Longrightarrow\; dp_1 = -(i\omega l\,dx + r_\nu dx)U_1$$

$$dU_1 = -\frac{i\omega A dx}{\gamma p_0}\left[1+(\gamma-1)f_\kappa\right]p_1 + \frac{\left(f_\kappa - f_\nu\right)}{\left(1-f_\nu\right)(1-\Pr)}\frac{dT_0}{T_0}U_1$$

$$\Longrightarrow\; dU_1 = -(i\omega c\,dx + \frac{1}{r_k}dx)p_1 + g dx U_1$$



- Inertance l>0

$$l = \frac{\rho_0}{A}\frac{1-\Re e\{f_\nu\}}{\left|1-f_\nu\right|^2}$$

- Compliance

$$c = \frac{A}{\gamma p_0}(1+(\gamma-1)\Re e\{f_\kappa\})$$

- Viscous resistance $r_\nu > 0$

$$r_\nu = \frac{\omega\rho_0}{A}\frac{\Im m\{-f_\nu\}}{\left|1-f_\nu\right|^2}$$

- Thermal resistance $r_k > 0$

$$\frac{1}{r_\kappa} = \frac{\gamma-1}{\gamma}\frac{\omega 1 \Im m\{-f_\nu\}}{p_0}$$

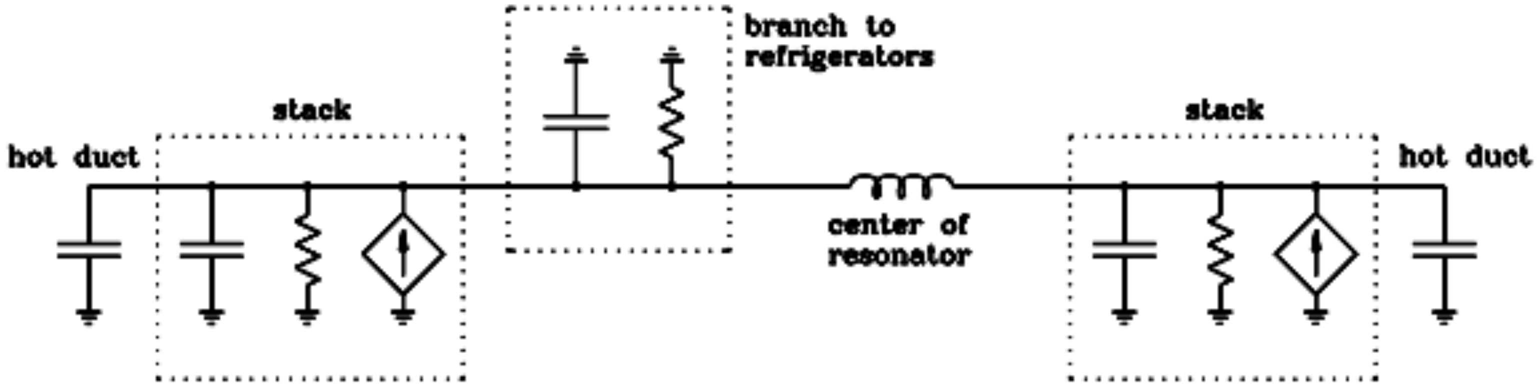


Figure 4.17: Schematic impedance diagram for the standing-wave engine example.

G.W. Swift, « Thermoacoustics: a unifying perspective for some engines and refrigerators, 2002

DELTA EC

⇨ Code for simulation and design of thermoacoustic or acoustic systems.

- 1993 : DELTA E, Ward et Swift, Los Alamos
- 2007 : new version of DELTA EC, Ward et Swift, Los Alamos

⇨ Free uploading, for Windows or Macintosh (source code unavailable)

http://www.lanl.gov/thermoacoustics/DeltaEC.html

- System :
  - Standing- or travelling-wave thermoacoustic engine or refrigerator,.
  - Any other 1D acoustic system
- User has the choice of gas (Air, He, Ar...) or mixture of working gas, of the stack material, of mean pressure and mean temperature.
- The system is user-defined as an assembly of segments (~20 different types) : stack (plates, circular or rectangular pores, pin arrays) regenerator, tube, convergent, transducer, pressure losses.

Numerical solution of systems of 1D-linearized wave equations, in permanent regime (e^{iwt}).

For each segment : solve for :

$$
\begin{aligned}
\frac{dp_1}{dx} &= \mathcal{F}_{\text{momentum}}(\text{local and global parameters and variables}),\\
\frac{dU_1}{dx} &= \mathcal{F}_{\text{continuity}}(\text{local and global parameters and variables}),\\
\frac{dT_m}{dx} &= \mathcal{F}_{x\text{-energy}}(\text{local and global parameters and variables}),\\
\frac{dT_{\text{solid}}}{dx} &= \mathcal{F}_{\text{solid}}(\text{local and global parameters and variables}),\\
\frac{d\dot{H}_{\text{tot}}}{dx} &= \mathcal{F}_{\text{lateral energy}}(\text{local and global parameters and variables}),\\
\frac{dp_{2,0,HL}}{dx} &= \mathcal{F}_{\text{head loss}}(\text{local and global parameters and variables}),\\
\frac{dn_L}{dx} &= \mathcal{F}_{\text{mix sep}}(\text{local and global parameters and variables}).
\end{aligned}
$$

Global variables: gas, mean temperature, mean pressure, frequency
Local variables: p_1, U_1, T_m, H, geometry
+ continuity of p_1 and u_1, T_m, $p_{2,0,HL}$ and n_L between each segment.

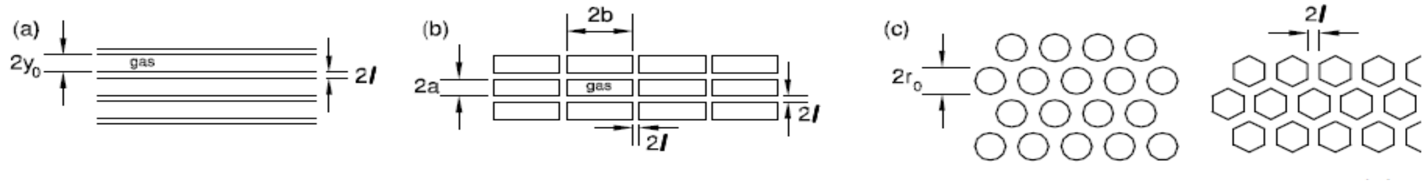
DUCT

$$\frac{dp_1}{dx} = -\frac{i\omega\rho_m}{(1-f_\nu)A}U_1,$$
$$\frac{dU_1}{dx} = -\frac{iA\omega}{\rho_m a^2}\left(1+\frac{\gamma-1}{1+\epsilon_s}f_\kappa\right)p_1,$$

+ continuity of p_1 and u_1 between segments

Stack/Regenerator STK*

(DeltaEC Users Guide, Ward 2008)



Pressure propagates according to Rott's wave equation, written in the form

$$\frac{dp_1}{dx} = -\frac{i\omega\rho_m}{(1-f_\nu)A_{\text{gas}}}U_1, \tag{9.51}$$
$$\frac{dU_1}{dx} = -\frac{i\omega A_{\text{gas}}}{\rho_m a^2}\left(1+\frac{(\gamma-1)f_\kappa}{1+\epsilon_s}\right)p_1 + \frac{\beta(f_\kappa-f_\nu)}{(1-f_\nu)(1-\sigma)(1+\epsilon_s)}\frac{dT_m}{dx}U_1, \tag{9.52}$$

subject to the condition that the total energy flow $\dot{H}_{2,k}$ is independent of x, which imposes the following condition on $T_m(x)$:

$$\frac{dT_m}{dx} = \frac{\dot{H}_{2,k} - \frac{1}{2}\mathrm{Re}\left[p_1\tilde{U}_1\left(1-\frac{T_m\beta(f_\kappa-\tilde{f}_\nu)}{(1+\epsilon_s)(1+\sigma)(1-\tilde{f}_\nu)}\right)\right]}{\frac{\rho_m c_p|U_1|^2}{2\omega A_{\text{gas}}(1-\sigma)|1-f_\nu|^2}\mathrm{Im}\left[\tilde{f}_\nu + \frac{(f_\kappa-\tilde{f}_\nu)(1+\epsilon_s f_\nu/f_\kappa)}{(1+\epsilon_s)(1+\sigma)}\right] - A_{\text{gas}}k - A_{\text{solid}}k_s}. \tag{9.53}$$

+ continuity of $p_{1,}u_{1,}$Tm, between segments

Numerical procedure, starting from initial segment

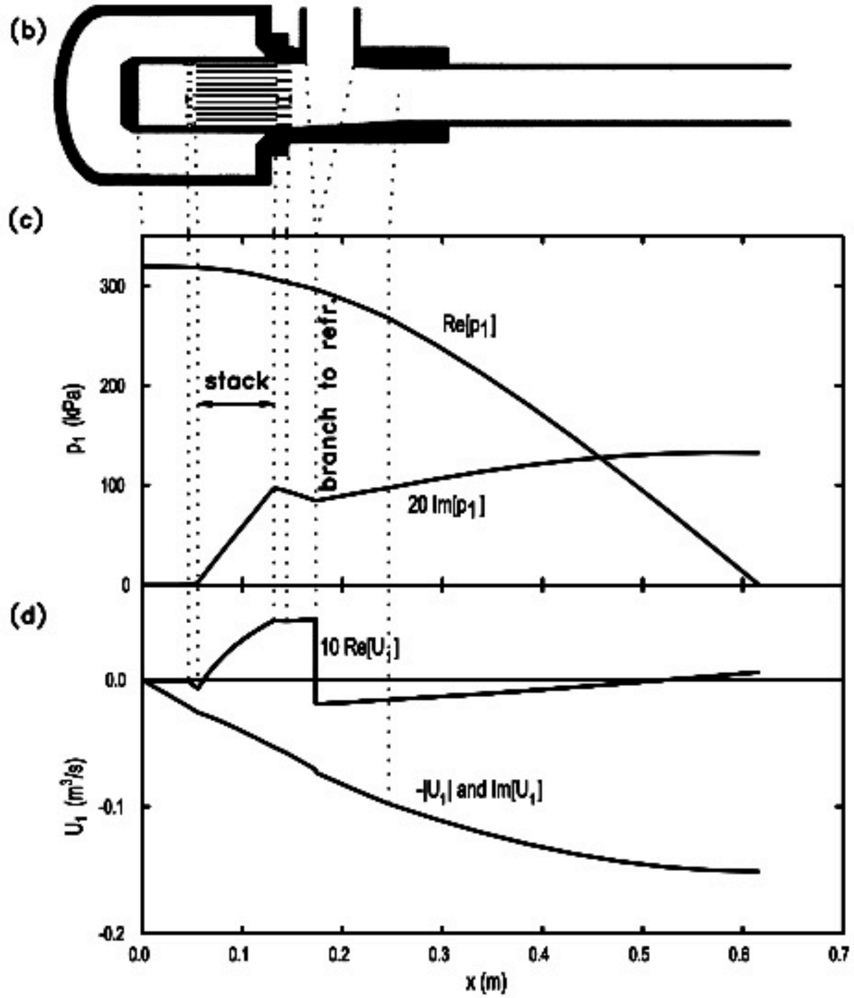
- Shooting method
  - N unknown parameters (« Guesses ») to be determined *(f, p1, U1, ..)*
  - N « Targets » are fixed parameter values

The unknown parameter values are adjusted until all targets are reached.

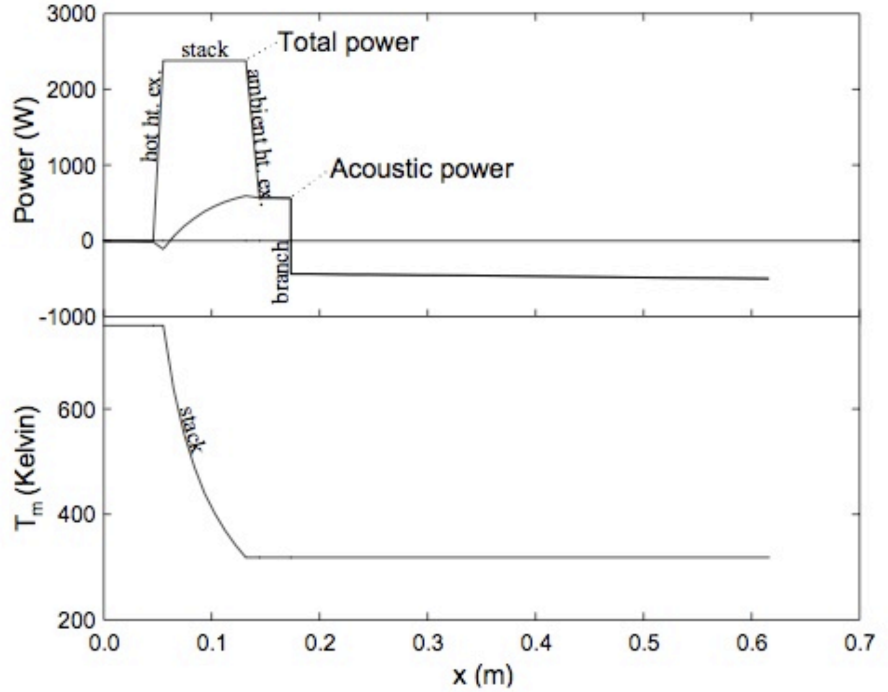
Newton-Raphson method (guesses have to be close to the solution)

Non linear effects included:

- Singular pressure losses at each segment, section change, angle, ….
- Regular pressure losses
- Mean flow
- 2D or 3D effects, Rayleigh streaming, jet driven streaming can be treated by adding pressure losses or a thermal load near at heat exchangers.



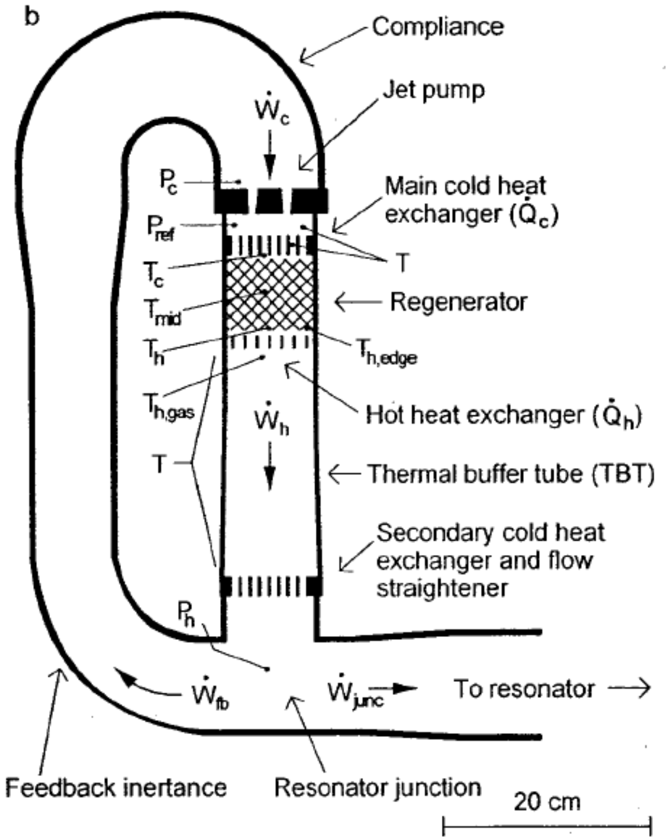
4.18: Results of numerical integration of the momentum and continuity equations for
nding-wave engine example first introduced in Figs. 1.8 and 1.9. (a) Relevant portions
e impedance diagram. (b) Scale drawing of apparatus. (c) p_1. (d) U_1. The numerical
tion unrealistically assigned the entire impedance of the "branch to refrigerator" at
nt $x = 0.18$ m, causing unrealistic discontinuities there.

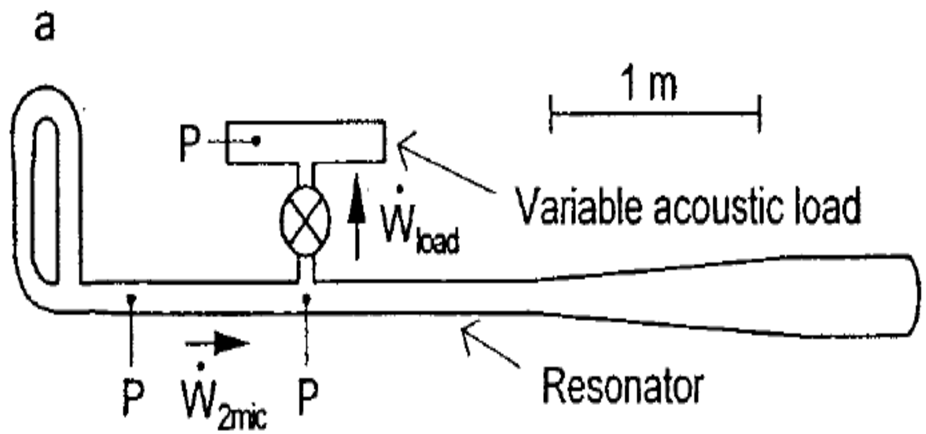


5.7: Acoustic power $\dot{E}_2$, total power $\dot{H}_2$, and mean temperature T_m as
ı x for the standing-wave engine, under the same conditions as Fig. 4
ger" is abbreviated "ht. ex."

Example: Standing-wave engine
(Swift, 1999)

Example : TASHE Backhaus / Swift 1999





- Pm = 30 bar
- Helium

(DeltaEC Users Guide, Ward 2008)

CRISTA, MARGO , TADESIGN

⇨ Based of linear theory of thermoacoustics based on Rott's equations.

Developped in France CNRS / Hekyom Society

⇨ CRISTA (similar to DELTAEC)

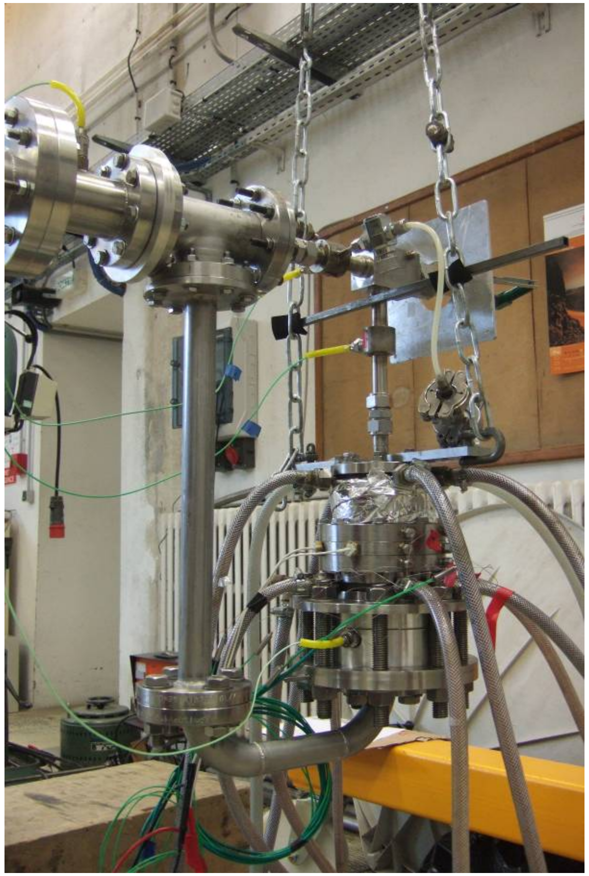
- Equations solved with a Runge-Kutta scheme of 4th order with an adaptative step.
- Based on shooting method
- Coupling engine/load (shooting method on each part on acoustics variables, engine temperature and frequency)
- Different acoustic architectures, including loops, derivation elements, multiples cells

⇨ TADESIGN (similar approach to DELTAEC)

- Design of thermoacoustic engines, refrigerators, acoustic amplifiers, etc

⇨ MARGO (linear stability code)

Prediction of the theresholds stability conditions (mean pressure, temperature gradient…) for each acoustic mode



Example: pulse tube refrigerator, coupled with standing wave generator, Hekyom 2007:

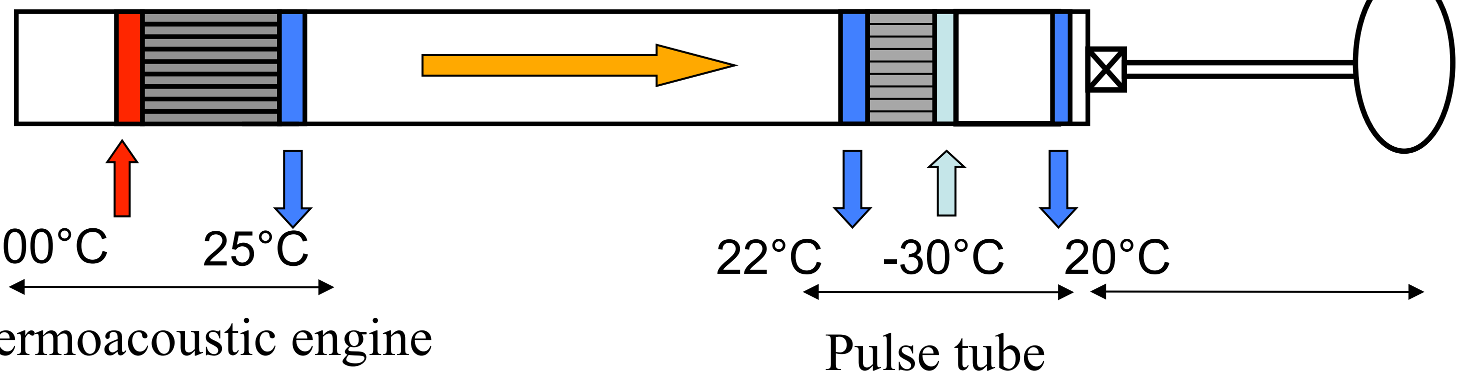
loop architecture

$P_m = 26.4$ bar, f = 68.1 Hz

$p_{1,tgp} = 1.351$ bar

$T_{condenseur} = -108°C$

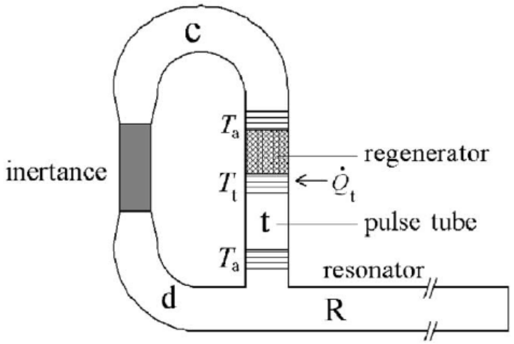
RLC architecture:



Time dependent model derived from linear theory

A.T.A.M. de Waele, Journal of Sound and Vibration 325 (2009)

Set of differential equations describing dynamics of individual components
Condensed into a single ODE -> time dependence of all dynamic variables.
Analytical expressions are obtained for damping coefficient, oscillation frequency, and onset temperature that allows stable oscillations.
Transient effects are obtained with numerical integration of dynamic equations.



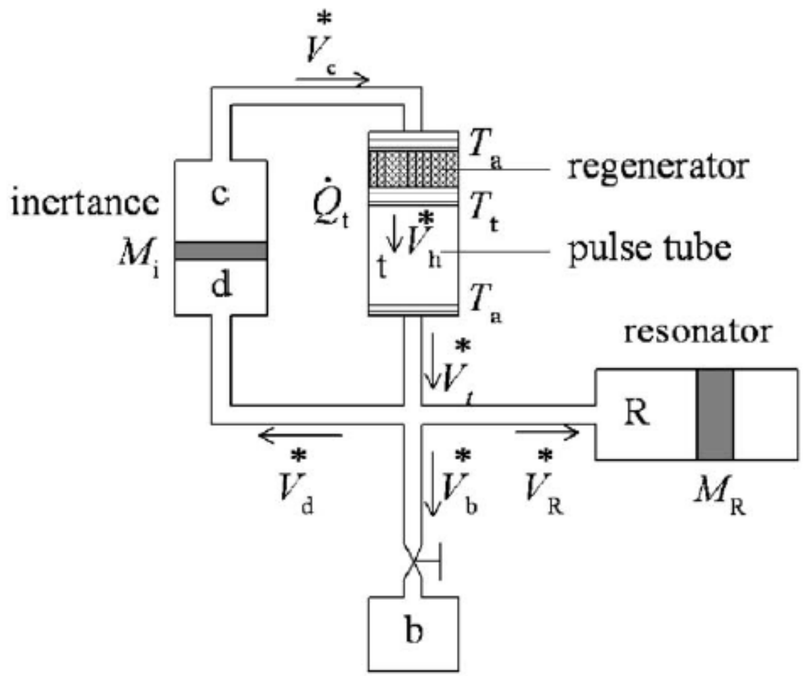
$$\frac{d^4\delta p_t}{dt^4} + a_3\frac{d^3\delta p_t}{dt^3} + a_2\frac{d^2\delta p_t}{dt^2} + a_1\frac{d\delta p_t}{dt} + a_0\delta p_t = 0$$

$$a_3 = w_eC_o + w_cC_r + \tau_t w_eC_r,$$

$$a_2 = a_Rw_e + a_iw_c + a_iw_e + w_cw_eC_rC_o,$$

$$a_1 = w_cw_e(C_ra_R + C_oa_i),$$

$$a_0 = w_cw_ea_Ra_i.$$



- **The linear theory allows to:**

  - compute the acoustic field and performance for a given device.
  - predict the instability threshold through a linear stability analysis, for a given parameter set.
  - choose the fixed / variable parameters in 1D codes

- **Limits:**

  - simulations are done for one frequency only (one mode)
  - only the periodic regime is described, no transients
  - no hydrodynamic or acoustic nonlinearity
  - multidimensional effects partially accounted for

Nonlinear 1D ans 2D models for thermoacoustic engines

Karpov and Prosperetti work, 1997-2000

- quasi 1D non linear model integrated with a TVD (total variation diminishing) scheme: describe the growth and saturation of oscillations
- weakly non linear model fort the thermoacoustic instability developped in the time domain (mutiple time scales): analytical results in initial growth, non linear evolution, saturation

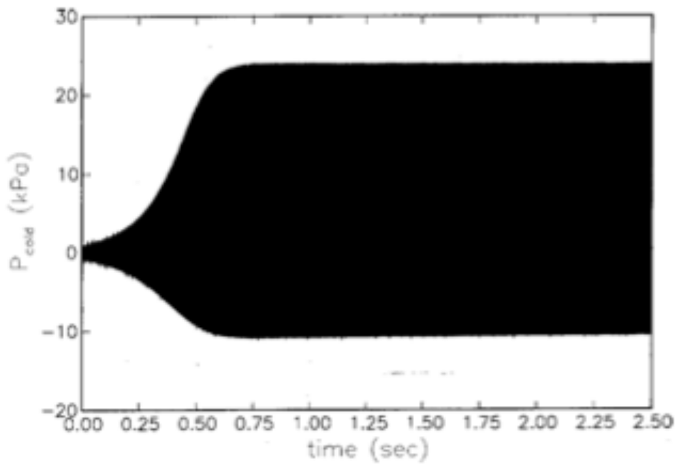


FIG. 10. Transient behavior of the pressure at the cold end of the tube for a prime mover with a temperature difference $T_H - T_C = 368$ K described by Atchley *et al.* (1990b).

Hamilton et al. 2002

- Non-linear two dimensional model based on boundary layer approximation
- Finite differences with a two-step Lax-Wendroff scheme
- Variable geometry

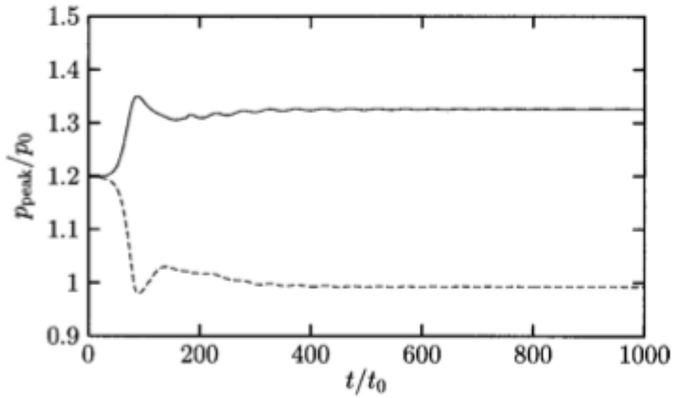


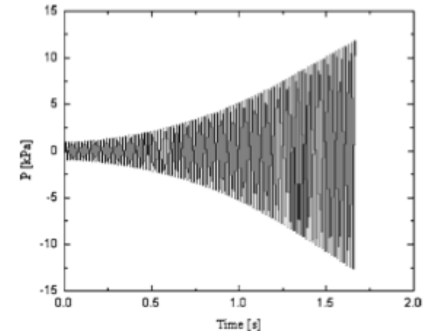
FIG. 4. Onset of oscillations and transition to steady state in an engine with constant cross section. Solid line is peak positive pressure in the waveform at the left end of the engine, dashed line is the corresponding peak negative pressure.

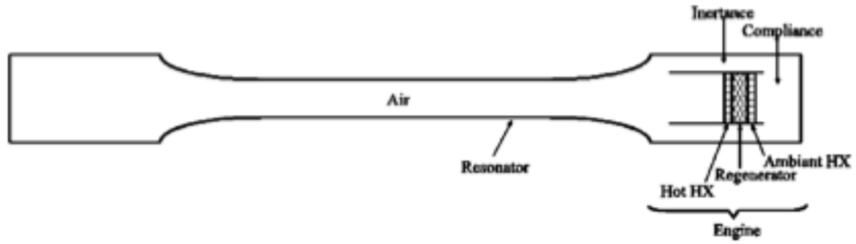
Compressible Navier Stokes equations and associated numerical simulations

- No 3D simulation of full Navier Stokes equations available in the litterature (too time consuming)

- Simulations of 2D axisymmetric NS equations (resonator)+porous medium model equations (regenerator, heat exchangers)

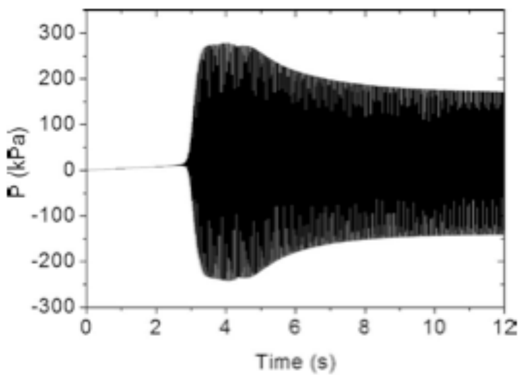
Lycklama et al. 2005 (finite volumes with **CFX,** travelling wave thermoacoustic engine)
Wave amplification-on set of oscillations





1-4 Yu *et al.*

G. Y. Yu et al. 2007 (finite volumes with **Fluent,** Thermoacoustic Stirling heat engine ,TASHE)
Onset temperature, wave amplification and saturation



Compressible Navier Stokes equations and associated numerical simulations

- Full NS equations, time simulation, Finite Volumes with unstructured mesh, flux reconstruction method, fully-explicit, third-order Runge-Kutta scheme + parallelized (MPI) protocol.

Scalo et al, AIAA 2013 : 2D axisymmetric numerical simulation of a model traveling-wave TASHE (double-Helmholtz resonator with the engine module at one end)

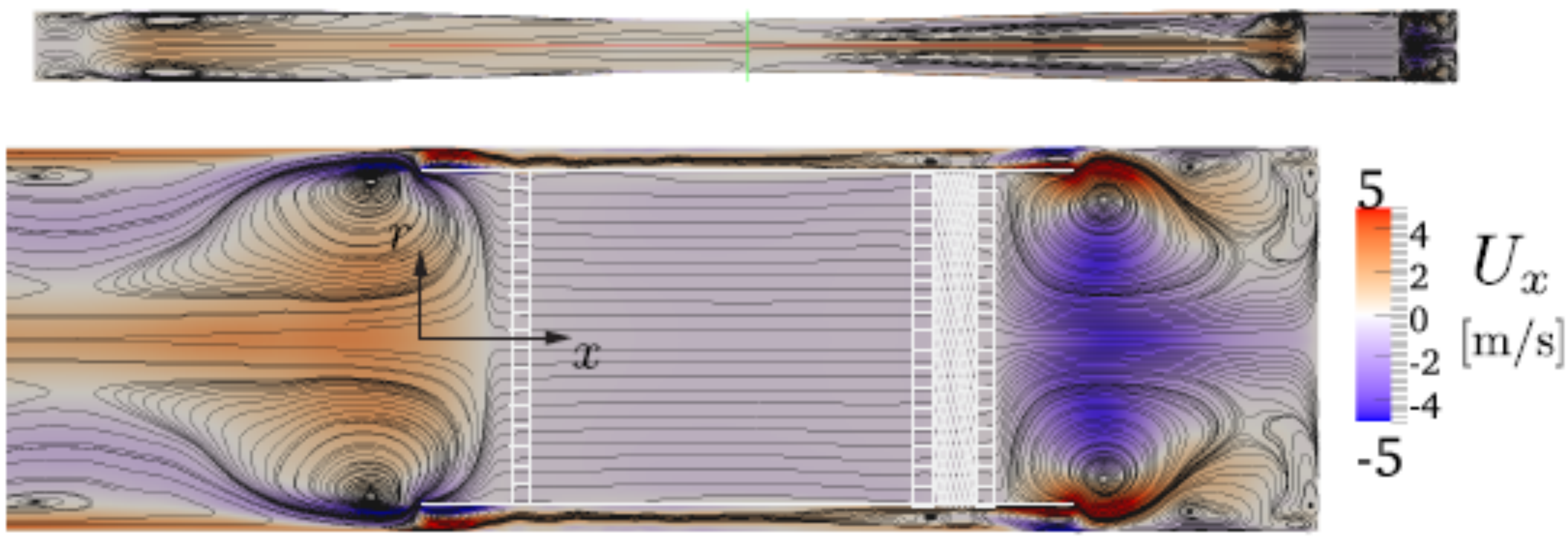


Figure 11. Contours of axial component of time-averaged streaming velocity (U_x) and corresponding streamlines. Full-scale visualization (top) and zoom on the right end (bottom). Results have been mirrored about the centerline for illustrative purposes.

Hybrid models. Low Mach number approximations.
2D simulations

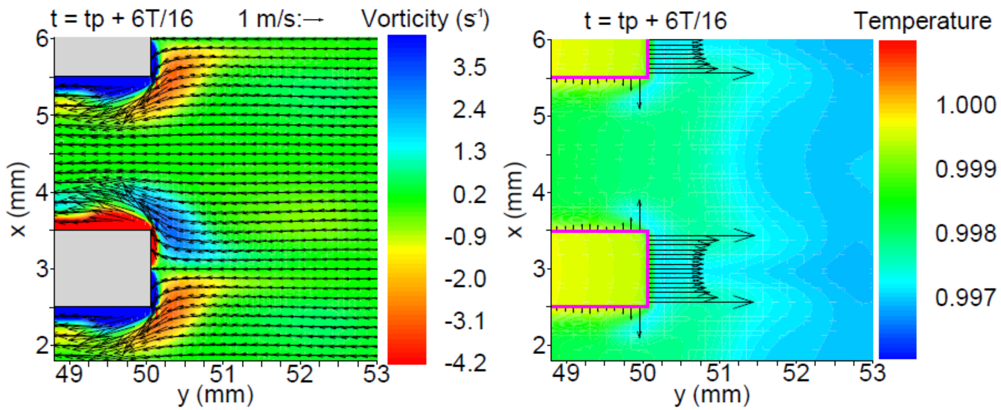
Multiple spatial scale analysis / given acoustic time scale τ

$$c \sim \frac{L_{res}}{\tau} \qquad U \sim \frac{L_{stack}}{\tau} \qquad M \sim \frac{U}{c} = \frac{L_{stack}}{L_{res}} << 1$$

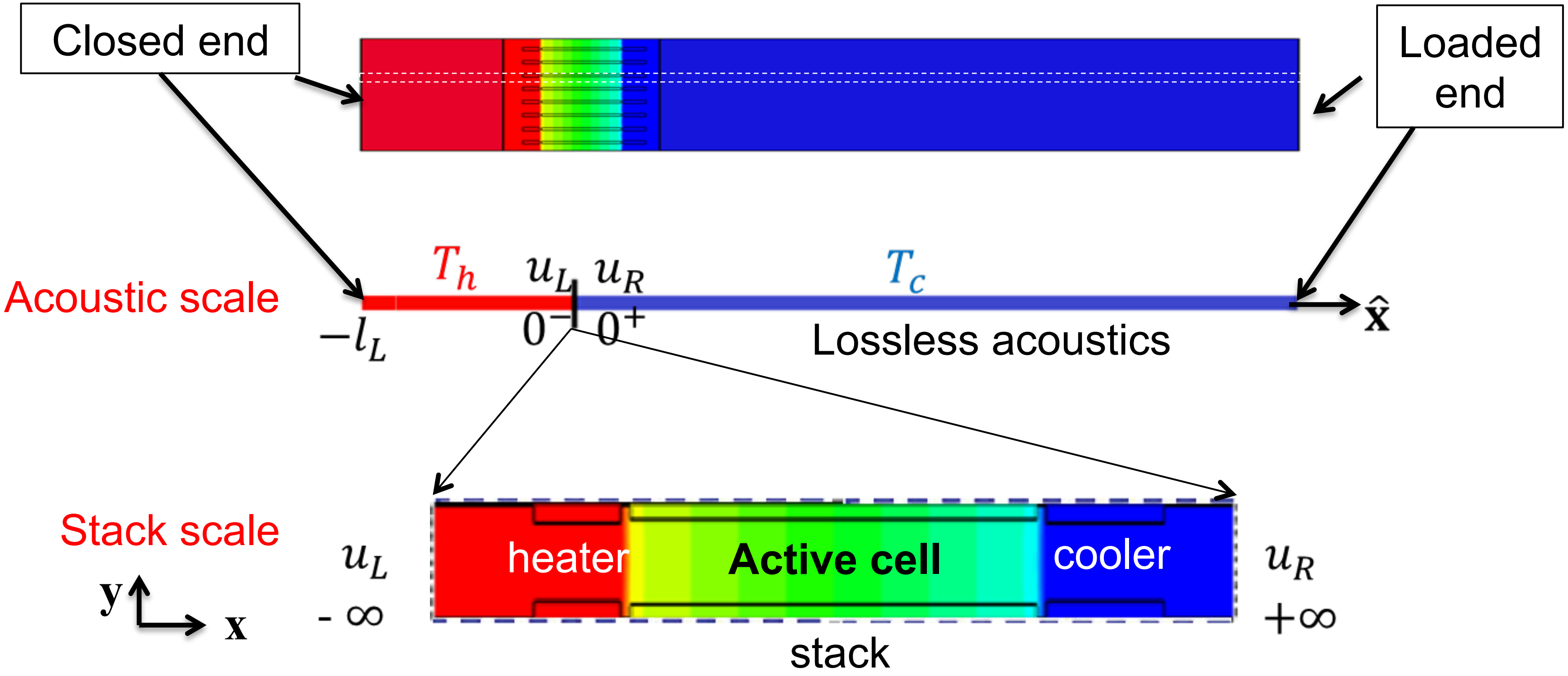
Short stack approximation leads to low Mach approximation

→ Linear acoustics in the resonator
→ Dynamically incompressible Navier-Stokes equations in stack with leading order variations of density

Applications :
- Worlikar and Knio, 1997-1999 (thermoacoustic refrigerators)
- Besnoin, Blanc Benon, Knio 2002
- Duthil, 2004
- Hireche et al., 2010



Ex: Low Mach number of SW thermoacoustic engine

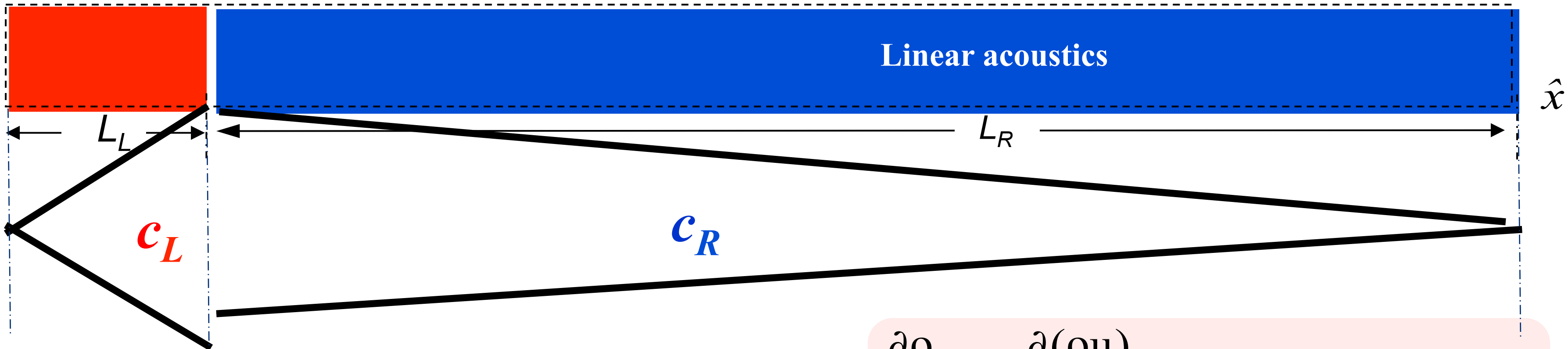


- Resonator: 1D linear acoustics → 1D analytical solution
- Active cell: Low Mach approximation of Navier-Stokes → Numerical resolution 2D
- Coupling : Matched asymptotic expansions
  Main hypothesis: short stack (acoustically compact active cell)

➢ Unsteady simulation starting from thermal equilibrium and a small velocity perturbation in the resonator.

I. Resonator

- Spatial scale (acoustic) : $L_{res} = O$ (wavelength)
- Low Mach approximation of Euler equations: linear acoustics
- Pressure fluctuations (in time and space) of O(M)
- Arbitrary wave form and frequency



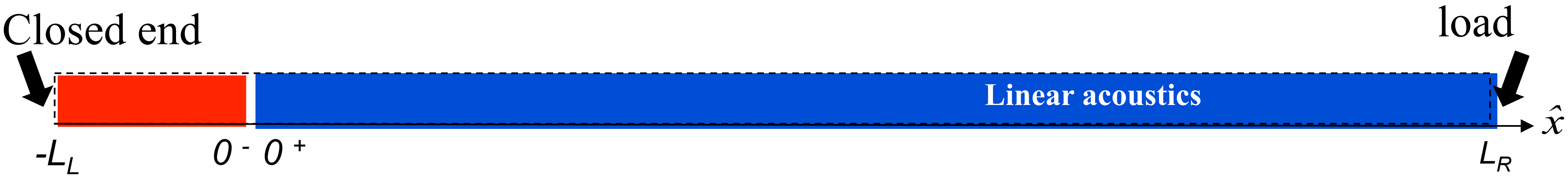
$$\frac{\partial \rho}{\partial t} + M \frac{\partial (\rho u)}{\partial \hat{x}} = 0$$

$$\rho \frac{\partial u}{\partial t} + M\rho u \frac{\partial u}{\partial \hat{x}} = -\frac{1}{\gamma M} \frac{\partial p}{\partial \hat{x}}$$

$$\rho \frac{\partial T}{\partial t} + M \frac{\partial (\rho u T)}{\partial \hat{x}} = \Gamma \frac{\partial p}{\partial t} + M\Gamma u \frac{\partial p}{\partial \hat{x}}$$

Reference

$$L_{ref} = L_{res}, \; p_{ref} = p_0, \; T_{ref} = T_0$$



Asymptotic developments in powers of M

$$\frac{\partial \rho^{(1)}}{\partial t} + \frac{\partial(\rho^{(0)} u^{(0)})}{\partial \hat{x}} = 0$$

$$\rho^{(0)} \frac{\partial u^{(0)}}{\partial t} + \frac{1}{\gamma}\frac{\partial p^{(1)}}{\partial \hat{x}} = 0$$

$$\frac{\partial s^{(0)}}{\partial t} + u^{(0)} \frac{\partial s^{(0)}}{\partial \hat{x}} = 0$$

$$p^{(0)} = \rho^{(0)} T^{(0)} = 1$$

Analytical solution (Riemann invariants)

$$\frac{\partial}{\partial t}\left(\gamma u^{(0)} \pm \sqrt{T^{(0)}}\, p^{(1)}\right) + \sqrt{T^{(0)}}\, \frac{\partial}{\partial \hat{x}}\left(\gamma u^{(0)} \pm \sqrt{T^{(0)}}\, p^{(1)}\right) = 0$$

$$R = \gamma u^{(0)} + \sqrt{T^{(0)}}\, p^{(1)} \ ; \ L = \gamma u^{(0)} - \sqrt{T^{(0)}}\, p^{(1)}$$

Closed end: $u^{(0)}(\hat{x} = -L_L) = 0$

$$R(0^-, t) = \gamma\, u(0^-, t) + \sqrt{T_{hot}}\ p^{(1)}(0^-, t) = \text{-}L(0^-, t_{back}^{left}) \qquad t_{back}^{left} = t - \frac{2 \cdot L_L}{\sqrt{T_{hot}}}$$

Loaded end: $p^{(1)}(\hat{x} = L_R) = f u^{(0)}(\hat{x} = L_R)$

$$L(0^+, t) = \gamma\, u(0^+) - \sqrt{T_{cold}}\ p^{(1)}(0^+) = \frac{\gamma - f}{\gamma + f} R(0^+, t_{back}^{right}) \qquad t_{back}^{right} = t - \frac{2 \cdot L_R}{\sqrt{T_{cold}}}$$

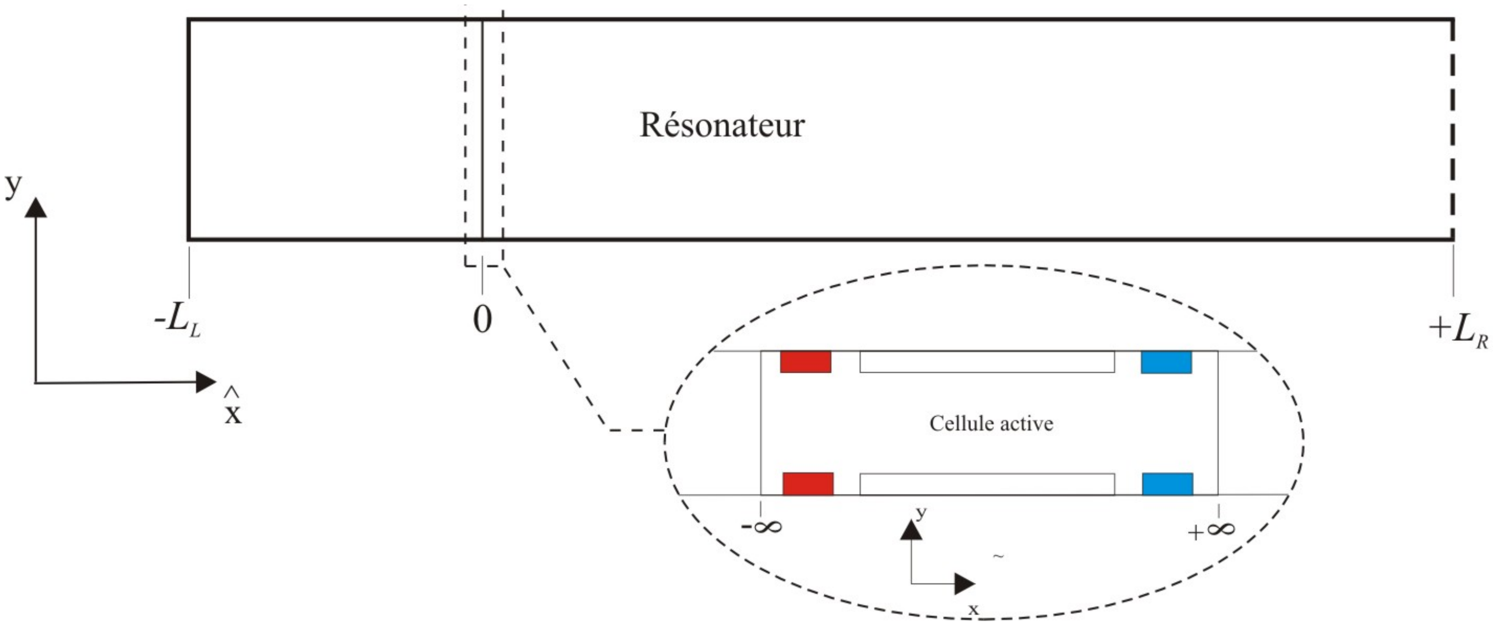
$$Z = \frac{\gamma - f}{\gamma + f}$$

$Z \to -1$ Open end

$Z \to +1$ Closed end

$Z \to 0$ Non reflecting wave

II. Heat exchangers/Stack (active cell)



• Hypotheses:

Spatial scale: L_{stack} = O(acoustic displacement) (interior scale)
Navier-Stokes equations / low Mach approximation + heat conduction in solids

Nonlinear "Incompressible" model with heat transfer

• Dynamic pressure correction of $O(M^2)$, superimposed to uniform spatial fluctuations (of dominant order)

• 2D geometry, potentially complex ⇨ **2D Numerical solution**

$$L_{ref} = L_{stack}\ ,\ p_{ref} = p_0\ ,\ T_{ref} = T_0$$

$$\frac{\partial \rho^{(0)}}{\partial t} + \nabla \cdot \rho^{(0)} \mathbf{V}^{(0)} = 0$$

$$p^{(0)} = p^{(0)}(t)$$

$$p^{(1)} = p^{(1)}(t)$$

Neglect gravity

$$\frac{\partial(\rho^{(0)}\mathbf{V}^{(0)})}{\partial t} + \nabla \cdot \left[\rho^{(0)}\mathbf{V}^{(0)} \otimes \mathbf{V}^{(0)}\right] = -\frac{1}{\gamma}\nabla p^{(2)} + \cancel{\frac{1}{Fr}(\rho^{(0)} - 1)\mathbf{e}_z} + \frac{1}{Re}\nabla \cdot \boldsymbol{\tau}^{(0)}$$

$$\tau = \mu^{(0)}\left[\nabla \mathbf{V}^{(0)} + (\nabla \mathbf{V}^{(0)})^t - \frac{2}{3}(\nabla \cdot \mathbf{V}^{(0)})\mathbf{I}\right]$$

$$\rho^{(0)} c_p^{(0)}\left[\frac{\partial T^{(0)}}{\partial t} + (\mathbf{V}^{(0)} \cdot \nabla) T^{(0)}\right] = \frac{1}{Pe}\nabla \cdot (k^{(0)} \nabla T^{(0)}) + \frac{\gamma - 1}{\gamma}\frac{dp^{(0)}}{dt}$$

$p^{(0)} = \rho^{(0)} T^{(0)}$ This formulation allows time variations of $p^{(0)}$

$$\frac{\partial T}{\partial t} = \frac{1}{Pe_s}\nabla^2 T$$

Heat conduction in plates

III. Matching:

- Two spatial scales: stack length $L_{stack} = O(U \times t_{ac})$
  resonator length $L_{res} = O(c \times t_{ac})$
- Length ratio $= O(U/c) = O(M) \rightarrow 0$

  Both solutions must match as $M \rightarrow 0$

  Interior sol. (position ∞) = Exterior Sol. (position of active cell)

$\rightarrow$ singular perturbation problem

Active cell transparent to pressure fluctuations but acts as a volume source.

$$\frac{\partial \rho^{(0)}}{\partial t} + \nabla \cdot \rho^{(0)} \mathbf{V}^{(0)} = 0$$

$$p^{(1)} = p^{(1)}(t)$$

$$\frac{\partial (\rho^{(0)} \mathbf{V}^{(0)})}{\partial t} + \nabla \cdot \left[\rho^{(0)} \mathbf{V}^{(0)} \otimes \mathbf{V}^{(0)} \right] = -\frac{1}{\gamma} \nabla p^{(2)} + \frac{1}{\mathrm{Re}} \nabla \cdot \boldsymbol{\tau}^{(0)}$$

$$\boldsymbol{\tau} = \mu^{(0)} \left[\nabla \mathbf{V}^{(0)} + (\nabla \mathbf{V}^{(0)})^{t} - \frac{2}{3} (\nabla \cdot \mathbf{V}^{(0)}) \mathbf{I} \right]$$

$$\rho^{(0)} c_p^{(0)} \left[\frac{\partial T^{(0)}}{\partial t} + (\mathbf{V}^{(0)} \cdot \nabla) T^{(0)} \right] = \frac{1}{\mathrm{Pe}} \nabla \cdot (k^{(0)} \nabla T^{(0)}) + \xcancel{\frac{\gamma - 1}{\gamma} \frac{d p^{(0)}}{d t}}$$

$$p^{(0)} = \rho^{(0)} T^{(0)} \boxed{= \text{ constant} = 1}$$

At each time step, coupling is ensured through solving 3 equations

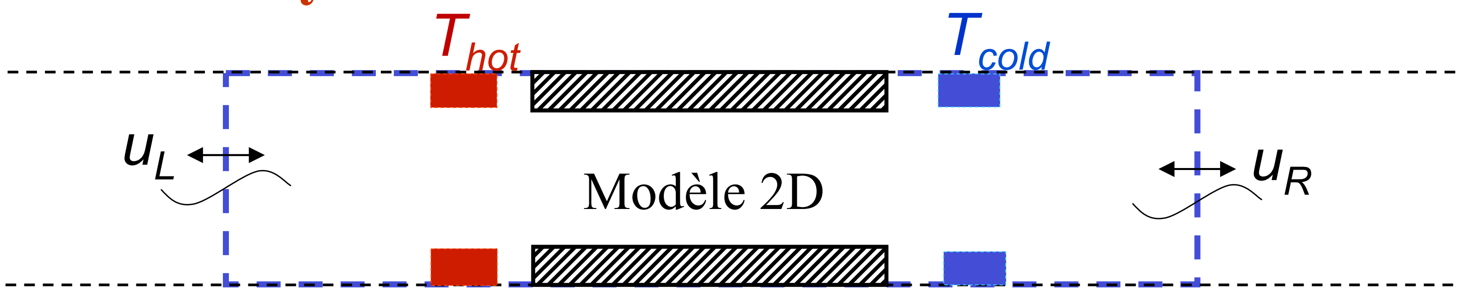
$$(\gamma\, u_L + \sqrt{T_{chaud}}\, p_{st}^{(1)})(t) = -(\gamma\, u_L - \sqrt{T_{chaud}}\; p_{st}^{(1)})(t_{back}^{left}) \qquad t_{back}^{left} = t - \frac{2 \cdot L_L}{L_T \sqrt{T_H}}$$

$$(\gamma\, u_R - \sqrt{T_{froid}}\, p_{st}^{(1)})(t) = Z(\gamma\, u_R + \sqrt{T_{froid}}\; p_{st}^{(1)})(t_{back}^{right}) \qquad t_{back}^{right} = t - \frac{2 \cdot L_R}{L_T \sqrt{T_C}}$$

$$H\left(u_L - u_R\right) + \frac{1}{Pe}\int_S \nabla T \cdot n dS = 0$$

$\rightarrow$ 3 eqns for 3 unknowns u_L, $u_{R\,,}\, p_{st}^{(1)}$

Boundary conditions and initial conditions



Adiabatic exterior boundaries (blue)

Slip on open horizontal boundaries

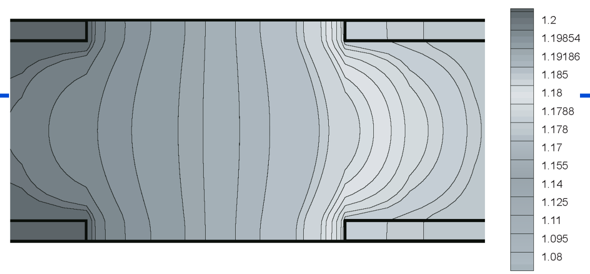
No-slip, continuity of temperature and heat flux at fluid/solid interfaces – stack and heat exchangers)

Temperature fixed on heat exchangers

Velocities u_L and u_R calculated using coupling with the resonator and energy balance.

Linear distribution of temperature between heat exchangers (gas and stack) OR steady heat conduction solution without flow

Resonator : random noise



Numerical method

- Finite volumes /second order
- Implicite scheme for viscous and diffusive terms
- Explicite scheme for convective terms
- Time integration: predictor-corrector scheme
- Fields calculated on the whole fluid/solid domain (use of a scalar function to differentiate between fluid and solid points)
- At each time step, solve for u_L, u_R et $p^{(1)}$ as functions of previous values accounting for (i) propagation in the resonator and (ii) energy balance.

Uniform mesh: 512x32 (coarse) to 2048x128 (fine)

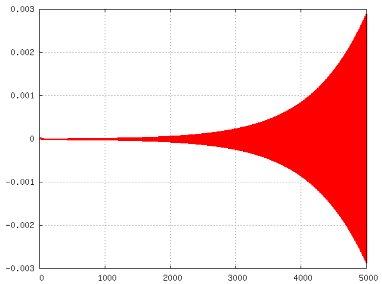
Minimum 500 time steps per reference acoustic period

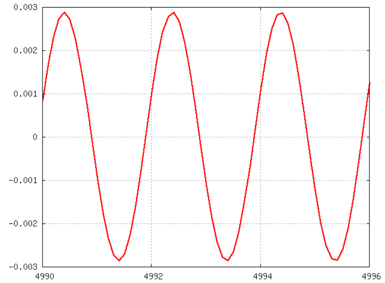
~ 0.5-2 hr CPU/run for the initial amplification phase

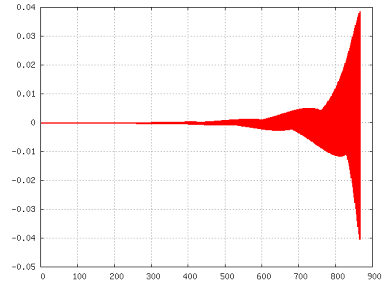
~ 50 hr CPU/run for obtaining wave saturation

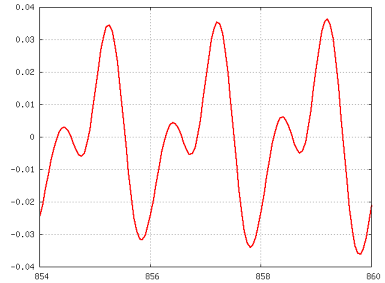
Results: thermoacoustic instability

The initial time signal of acoustic pressure contains multiple frequencies. It can be analyzed to determine the growth rate of each mode

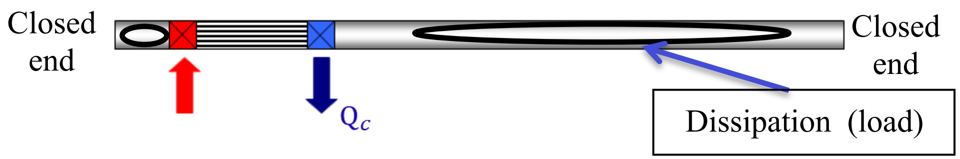


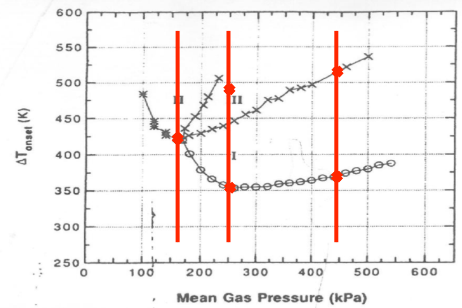






For each geometry and value of the load, the critical temperature can then be determined, as well as the associated most unstable mode.

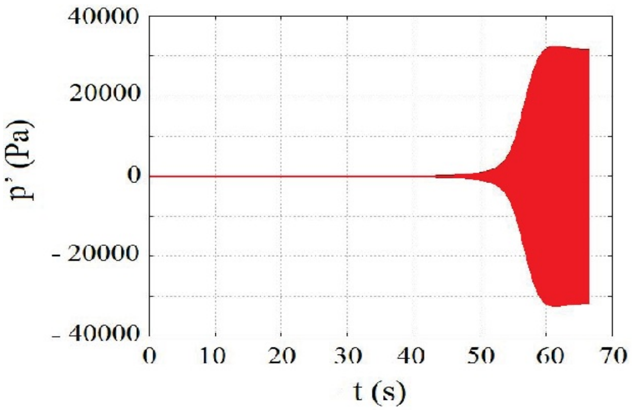




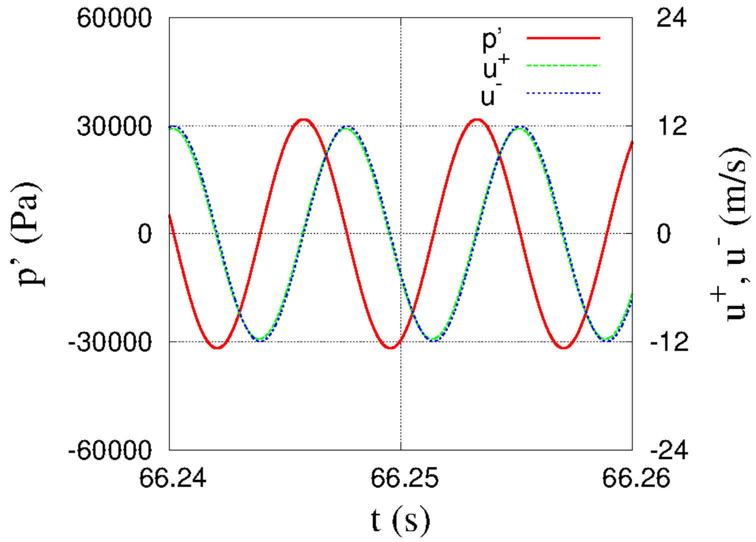
Results in agreement with Atchley et al, *JASA 91, 734-743 (1992)*
For 3 mean pressure values, a value of the load is determined such that the critical ΔT matches the experimental one.The corresponding mode is found to be the same.
For 440kPa, the transition is found correctly for mode I and mode II.

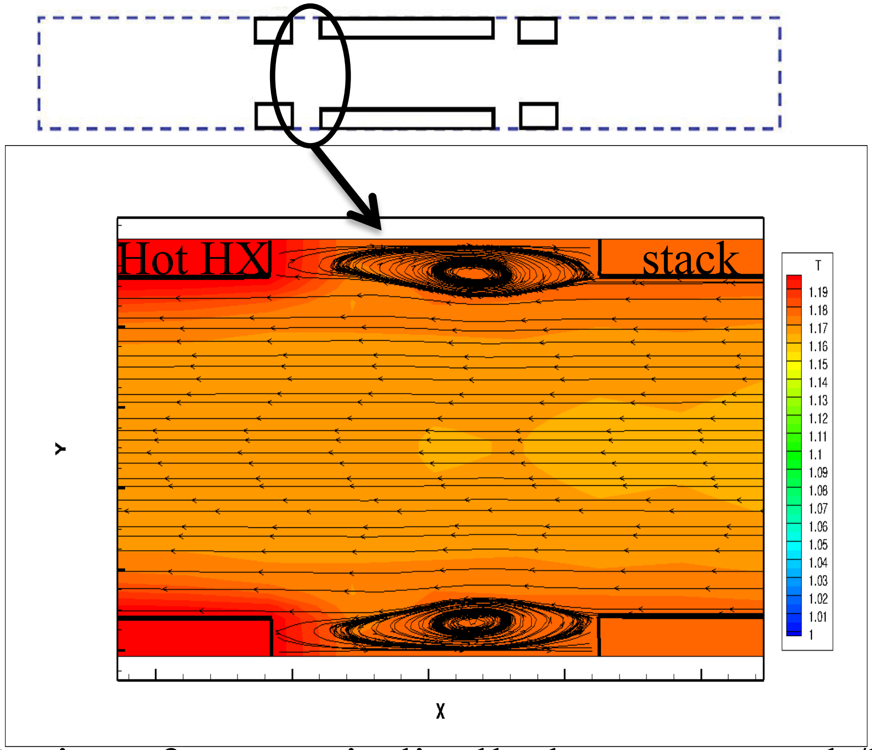
Value of the load representing the gas viscous dissipation giving the same critical temperature as the experiment

Results : Periodic regime



Adjust the load value and the temperature difference to obtain saturation





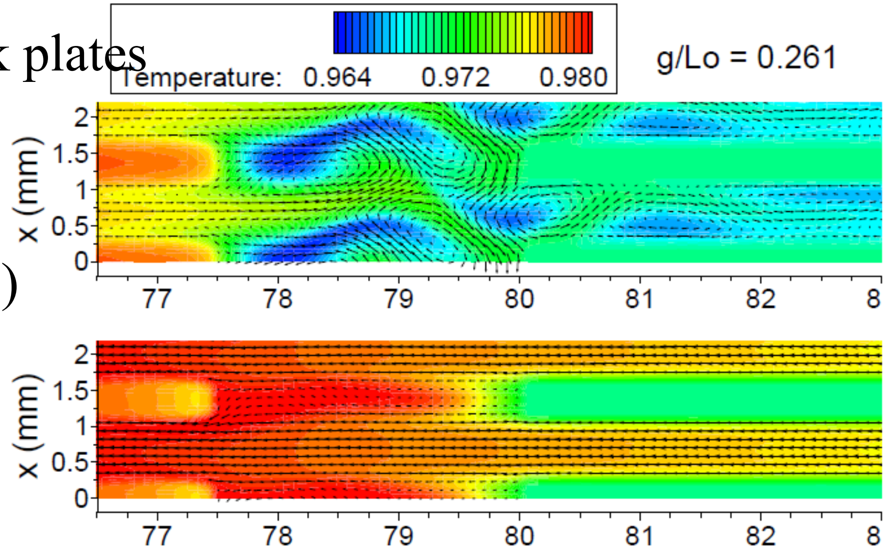
Vortices form periodically between stack/heat exchangers, and at entrance and exit of the heat-exchangers.

Estimate of acoustic power, heating power at the hot heat exchanger, efficiency :

$$P_{ac} = \frac{S}{\tilde{t}_{ac}} \int_{\tilde{t}_{ac}} \tilde{p}'_{st} \, \tilde{u} \, d\tilde{t} \qquad P_{ac} \approx 66W \qquad Q_h \approx 3.8kW \qquad \eta \approx 0.1\eta_c$$

Models and simulations for specific components of thermoacoustic devices

- Stacks: edge effects
  vortex generation at the end of stack plates
  low Mach number simulations
  compressible NS simulations
  (Marx and Blanc Benon 2004, 2005)
  thermal wave harmonics generation
  (Gusev et al. 2000, 2001,
  Marx and Blanc Benon 2005)



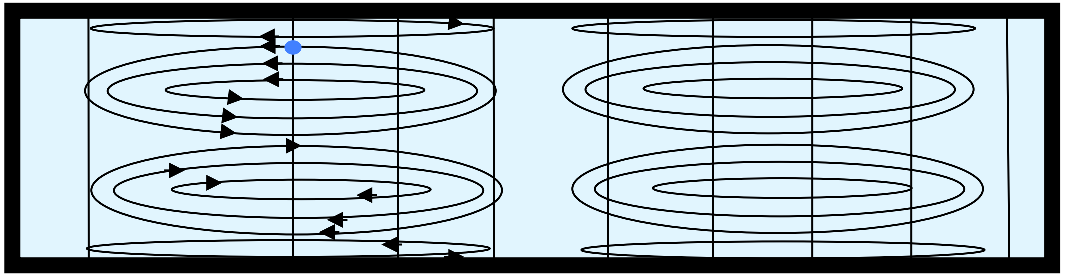
  Regenerators: porous media models; analytical models (Raspet et al. 1998, Swift 1996, 2002, Wilen 2001, Roh 2007) and hybrid lattice Boltzmann simulations (Jensen and Raspet 2009, 2010)

- Heat-exchangers models: analytical models (Mozurkewich, 1998,2001) and numerical simulations (Worlikar et al. 1998, Ishikawa and Mee, 2001, Besnoin and Knio 2004, Matveev et al. 2006, 2007)

- Resonators : **acoustic streaming linear and non linear regime**

Nonlinear streaming in a standing wave resonator

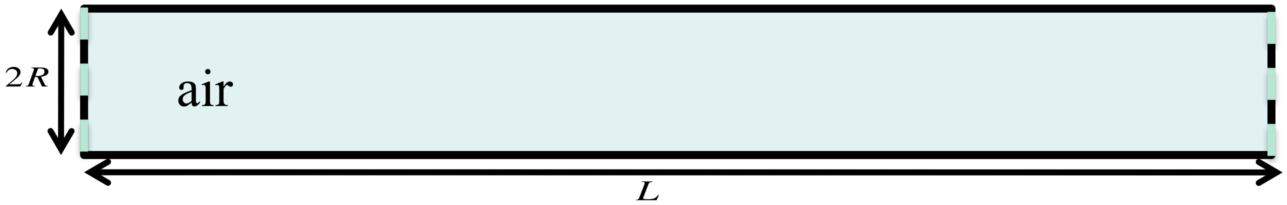
Streaming flow= 2^{nd} order mean flow produced by interaction acoustic wave/solid wall. Responsible for thermal loss in thermoacoustic devices.



Analytical models :
« slow » streaming: Rayleigh 1883, Schlichting 1932, Stuart 1966, Bailliet et al. 2001, Hamilton et al. 2003, Boumerfel et al. 2011
« fast » streaming : Menguy and Gilbert 2000

Experimental studies : Arroyo and Greated 1991, Campbell et al. 2000, Thompson and Atchley 2004, Moreau et al. 2008, Nabavi et al.2009

Numerical studies : Aktas and Farouk 2004, 2008,2009, Yano 2005 , Daru et al 2012 Boumerfel et al. 2011



Empty resonator, shaken axially with harmonic velocity, at resonance frequency
Full compressible Navier-Stokes equations, expressed in moving frame + source term (shaking)

$$\left\{\begin{array}{lcl} \dfrac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{v}) & = & 0 \\ \dfrac{\partial \rho\vec{v}}{\partial t} + \nabla\cdot(\rho\vec{v}\otimes\vec{v}) + \nabla p & = & \nabla\cdot(\bar{\bar{\tau}}) - \rho\dfrac{d\vec{V}}{dt} \\ \dfrac{\partial \rho E}{\partial t} + \nabla\cdot(\rho E\vec{v} + p\vec{v}) & = & \nabla\cdot(k\nabla T) + \nabla\cdot(\bar{\bar{\tau}}\vec{v}) - \rho\vec{v}\cdot\dfrac{d\vec{V}}{dt} \end{array}\right.$$

+ ideal gas law
+ no slip + fixed temperature on walls
+ symmetry conditions on vertical boundaries
+ Axisymmetric or cartesian geometry

Numerical method

Numerical schemes : explicit upwind finite differences (Daru & Tenaud 2004).

Convective terms: 3rd order in space/time Upwind scheme,

Diffusive terms : 2nd order centered scheme

Strang-splitting procedure used to obtain a 2nd order accuracy every 2 time steps

Scheme is implemented as a correction to the second order Mac-Cormack scheme.

$$w^*_{i,j} = w^n_{i,j} - \frac{\delta t}{r_{i,j}\delta r}[r_{i+(1/2),j}(f_{i+1,j} - f^v_{i+(1/2),j}) - r_{i-(1/2),j}(f_{i,j} - f^v_{i-(1/2),j})]^n + \delta t(h + h^s)^n_{i,j}$$

$$w^{**}_{i,j} = w^*_{i,j} - \frac{\delta t}{r_{i,j}\delta r}[r_{i+(1/2),j}(f_{i,j} - f^v_{i+(1/2),j}) - r_{i-(1/2),j}(f_{i-1,j} - f^v_{i-(1/2),j})]^*$$

$$w^{n+1}_{i,j} = \frac{1}{2}(w^n_{i,j} + w^{**}_{i,j}) + \frac{1}{r_{i,j}}(r_{i+(1/2),j}C^r_{i+(1/2),j} - r_{i-(1/2),j}C^r_{i-(1/2),j}).$$

Long physical transient periods (several hundreds of periods) to reach steady state

$\delta t = T / 6250$

Spatially uniform mesh /direction: 500x50 to 500x400

Results : simulation cases

Mach Number : $\mathrm{M} = \dfrac{U_{max}}{c_0} \in [0.005;\, 0.13]$

Nonlinear Reynolds number : $\mathrm{Re}_{NL} = \left(M \dfrac{R}{\delta_\nu} \right)^2 \in [0.04;\, 29]$ $\dfrac{R}{\delta_\nu} = 40$

Asymptotic model $\delta_\nu / R << 1, \; M << 1$

Streaming equations outside the boundary layer (Menguy & Gilbert, 2000) :

$$\begin{cases} \dfrac{\partial \mathrm{u}_{2s}}{\partial \mathrm{x}} + \dfrac{1}{\mathrm{r}} \dfrac{\partial}{\partial \mathrm{r}} (\mathrm{r v}_{2s}) = 0 \\ \dfrac{1}{\mathrm{r}} \dfrac{\partial}{\partial \mathrm{r}} (\mathrm{r} \dfrac{\partial \mathrm{u}_{2s}}{\partial \mathrm{r}}) = \boxed{\mathrm{Re}_{\mathrm{NL}}} (\mathrm{u}_{2s} \dfrac{\partial \mathrm{u}_{2s}}{\partial \mathrm{x}} + \mathrm{v}_{2s} \dfrac{\partial \mathrm{u}_{2s}}{\partial \mathrm{r}}) + \mathrm{f}(\mathrm{x}) \end{cases}$$

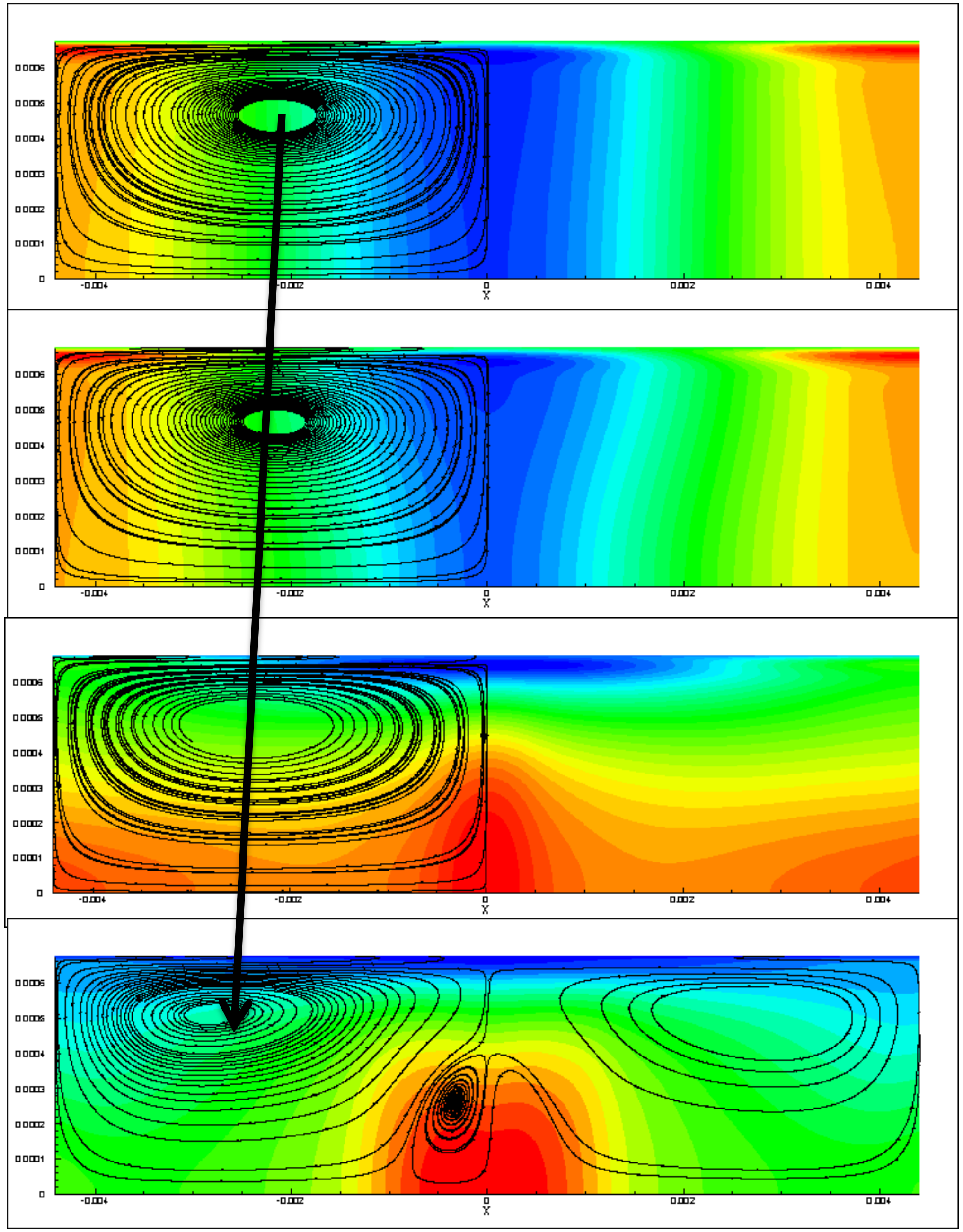
Slow Streaming
(regular) $\mathrm{Re}_{\mathrm{NL}} << 1$

$\tilde{\mathrm{u}} = \mathrm{c}_0 (\mathrm{M u_a} + \mathrm{M}^2 \mathrm{u_s})$

$\mathrm{f}(\mathrm{x}) =$ Acoustic terms (products of first order quantities) + 2nd order steady pressure gradient

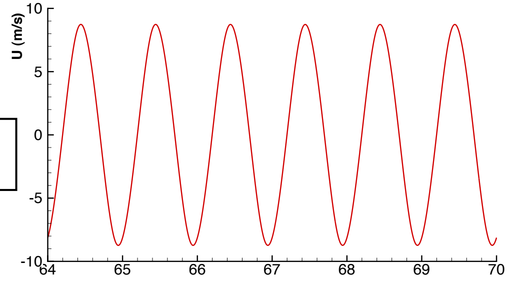
Fast Streaming
(irregular) $\mathrm{Re}_{\mathrm{NL}} \geq 1$

Example of results



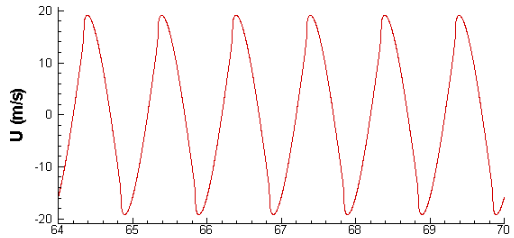
$Re_{NL}=1$

$\Delta T=0.002$



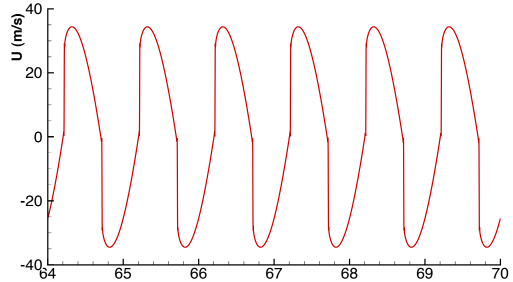
$Re_{NL}=5$

$\Delta T=0.26$



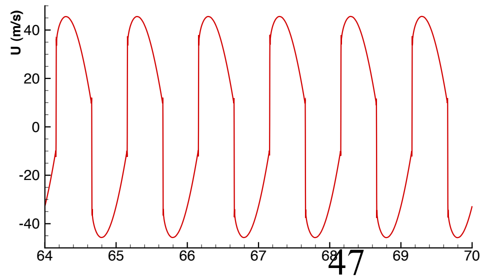
$Re_{NL}=16$

$\Delta T=1.57$



$Re_{NL}=29$

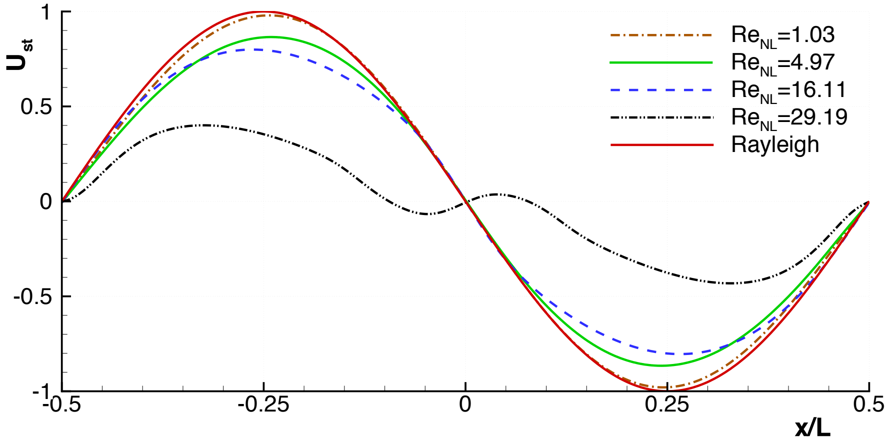
$\Delta T=9.55$



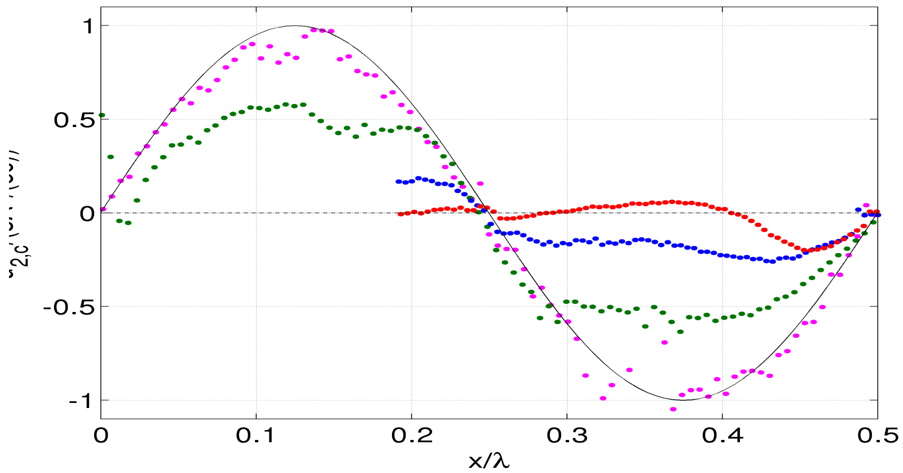
Comparison with experiments Reyt et. al 2013

Axial mean velocity U_{st}, normalized with $u_{Rayleigh} = 3U_{max}^2 / 8c_0$

Numerical results



Experimental result



⇨ Displacement of the maximum values towards velocity nodes

Similiar wave guides type: wide cylindrical resonators

Similar thermal wall boundary conditions (fixed temperature)

Different acoustics for $6<Re_N$:

Simulations: presence of shocks   Experience: monofrequency wave

Different dimensions

Simulations: $\lambda = 17.6\text{mm}, f = 20000\text{Hz}$ Experience: $\lambda = 1.42\text{m}, f = 240\text{Hz}$

Streaming dynamics in the non linear regime

- Numerical study of transition from slow acoustic streaming (linear regime) to fast acoustic streaming (non linear regime), in agreement with experiments

- Symmetry breakings for $Re_{NL} \geq 1$ and at $Re_{NL} \approx 30$ an additional vortex is generated on the central line, close to velocity antinode , again in agreement with experiments

- An intricate coupling between the mean temperature field and the streaming flow.

Conclusion

Incomplete review of current research in thermoacoustics:

Main observations:

- thermoacoustic devices are based on simple principles.... but the actual physics taking place is complex
- numerical codes used for design (DELTA-EC, CRISTA,...) and estimate of efficiency are based on a linear, periodic and 1D analysis when reality is very different.

Main current academic research (object : establish some models that can be included in numerical codes used for design)

- detailed models of stack/regenerators with complex geometry
- models/optimisation of heat exchangers : thermal and hydrodynamic edge effects , variable geometry
- fine description of coupling effects in: jet pumps, T, conical tubes
- coupling effects : acoustic source/ waveguide, thermoacoustic engine/ alternator, thermoacoustic engine/TGP
- fine description of nonlinear effects: acoustic streaming, NL propagation, turbulence