EIGENFUNCTIONS ON DISSIPATION OF MHD TURBULENT VORTICES

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Introduction. We consider the unsteady flow of an incompressible electrically conducting fluid in the presence of a uniform magnetic field. The fluid domain is a layer of uniform thickness h bounded by two parallel insulating walls at $z = 0$ and $z = h$, and the magnetic field $|\mathbf{B}| = B_0$ is oriented in the z-direction (Fig. 1). The physical properties of the fluid are such that the usual Reynolds number is much larger than unity, the magnetic Reynolds number is smaller than unity, and the Hartmann number $\mathrm{Ha} = \sqrt{\sigma/\rho\nu}\ B_0 h$ is much larger than unity, where ν stands for the kinematic viscosity, σ for electric the conductivity and ρ for the density. In such conditions, the fluid flow is highly turbulent and the induced magnetic field is negligible in comparison with the applied one. Such conditions are typical of liquid metal experiments performed at laboratory scale. This kind of MHD turbulence has been the subject of many theoretical investigations. Sommeria and Moreau [1] derived the necessary conditions for the turbulence to become quasi-two-dimensional and they established a model equation, which exhibits a linear damping term, taking into account both Joule and viscous dissipations within the Hartmann layer. Another relevant property to understand the dynamics of MHD turbulence, discovered by Davidson [2], is the invariance of the component of the angular momentum, which is parallel to the magnetic field. In complement to the diffusion mechanism proposed by Sommeria and Moreau [1], this invariance is also a key to the understanding of the tendency toward two-dimensionality of the vortices in the direction of the magnetic field. The purpose of this paper is to investigate the damping mechanisms and to derive the damping rates of the quasi-two-dimensional vortices, which are likely to be present. To achieve this, we disregard the non-linear energy transfer between the different classes of vortices and we only take into account the liner terms in the relevant equations. Such an analysis allows to isolate individual classes of vortices, each of them being characterized by two parameters, a wavenumber k_{2D} in the plane perpendicular to the magnetic field, and a mode number m associated with the k_z component of the wave vector (see below for the full definition of these parameters). This purely linear investigation yields the damping time scales of each class of vortices, but it cannot yield their relative intensities, which are controlled by the neglected non-linear effects.

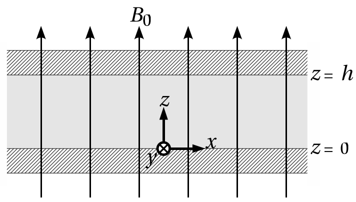


Fig. 1. Electrically conducting fluid layer bounded by two insulating walls.

1. Eigenvalue problem. The damping of vortices is analyzed with the following set of linear equations:

$$\frac{\partial \omega_z}{\partial t} = \nu\, \boldsymbol{\nabla}_{2\mathrm{D}}{}^2 \omega_z - \frac{\sigma B_0}{\rho}(B_0\, \omega_z - \boldsymbol{\nabla}_{2\mathrm{D}}{}^2 \phi), \qquad \boldsymbol{\nabla}^2 \phi = B_0\, \omega_z, \tag{1}$$

where $\boldsymbol{\nabla}_{2\mathrm{D}}{}^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ stands for the 2D Laplacian operator, ω_z for the vorticity component in the direction of the magnetic field, and ϕ for the electric potential.

Following the same procedure as Sommeria and Moreau [1], we replace the boundary conditions at the wall by matching conditions for the core flow at the edge of the thin Hartmann layers:

$$\partial\phi/\partial z \to -\sqrt{\rho\nu/\sigma}\ \omega_z \quad \text{for } z \to 0, \qquad \partial\phi/\partial z \to \sqrt{\rho\nu/\sigma}\ \omega_z \quad \text{for } z \to h. \tag{2}$$

The tangential components of the velocity u_x, u_y may accept some slip when $z \to 0$ or $z \to h$.

Let us now apply two-dimensional Fourier transformation to eq.(1) :

$$\frac{\partial \hat{\omega}}{\partial t} = -\left(\frac{k_{2\mathrm{D}} h}{\mathrm{Ha}}\right)^2 \frac{\hat{\omega}}{\tau_{\mathrm{J}}} - \frac{\hat{\omega}}{\tau_{\mathrm{J}}} - \frac{k_{2\mathrm{D}}{}^2 \hat{\phi}}{B_0\, \tau_{\mathrm{J}}}, \qquad \frac{\partial^2 \hat{\phi}}{\partial z^2} - k_{2\mathrm{D}}{}^2 \hat{\phi} = B_0\, \hat{\omega}, \tag{3}$$

where we denote $k_{2\mathrm{D}}{}^2 = k_x{}^2 + k_y{}^2$ and introduce the Joule time $\tau_{\mathrm{J}} = \rho/(\sigma B_0{}^2)$ [1, 3]. We express the solutions of linear equations (3) in the form $\hat{\omega} = f(z)\exp(-t/\tau)$,$\hat{\phi} = g(z)\exp(-t/\tau)$. In any actual turbulent flow, the vorticity may not be damped as the above exponential function, because there is nonlinear energy transfer. Nevertheless, we can estimate the dissipation in the nonlinear problem using the damping rate $1/\tau$ obtained from the linear problem. Substituting the above expressions into eqs.(3) yields the following equation for $f(z)$

$$\frac{\mathrm{d}^2 f}{\mathrm{d}z^2} + k_{2\mathrm{D}}{}^2 \alpha^2 f = 0, \qquad \text{where} \quad \alpha^2 = \left[1 + \left(\frac{k_{2\mathrm{D}} h}{\mathrm{Ha}}\right)^2 - \frac{\tau_{\mathrm{J}}}{\tau}\right]^{-1} - 1. \tag{4}$$

The boundary conditions (2) become

$$\frac{\mathrm{d}f}{\mathrm{d}z} \to \frac{k_{2\mathrm{D}}{}^2 h(1+\alpha^2)}{\mathrm{Ha}} f \text{ for } z \to 0, \ \ \frac{\mathrm{d}f}{\mathrm{d}z} \to -\frac{k_{2\mathrm{D}}{}^2 h(1+\alpha^2)}{\mathrm{Ha}} f \text{ for } z \to h. \tag{5}$$

Equation (4) and boundary conditions (5) form an eigenvalue problem. There is an infinite number of eigenvalues $k_{zm} \equiv \alpha_m k_{2\mathrm{D}}$ given by

$$\frac{k_{zm} h - m\pi}{2} = \mathrm{Arctan}\left(\frac{(k_{2\mathrm{D}}{}^2 + k_{zm}{}^2)h}{k_{zm}\mathrm{Ha}}\right) \qquad (m = 0, 1, 2, 3, \cdots). \tag{6}$$

Let us now call the integer m the mode number. The eigenvalues k_{zm} exist in the interval $m\pi < k_{zm} h < (m+1)\pi$ and, as a consequence, the eigenfunctions

$$f_m = \cos[k_{zm}(z - h/2)], \quad (m = 0, 2, 4, \cdots) \qquad f_m = \sin[k_{zm}(z - h/2)] \quad (m = 1, 3, 5, \cdots) \tag{7}$$

have m zeros in the gap $0 < z < h$. Finally, the following damping time is obtained in terms of the wavenumber $k_{2\mathrm{D}}$ and of the mode number m:

$$\tau_m = \tau_{\mathrm{J}} \left[\frac{k_{zm}{}^2}{k_{2\mathrm{D}}{}^2 + k_{zm}{}^2} + \left(\frac{k_{2\mathrm{D}} h}{\mathrm{Ha}}\right)^2\right]^{-1}. \tag{8}$$

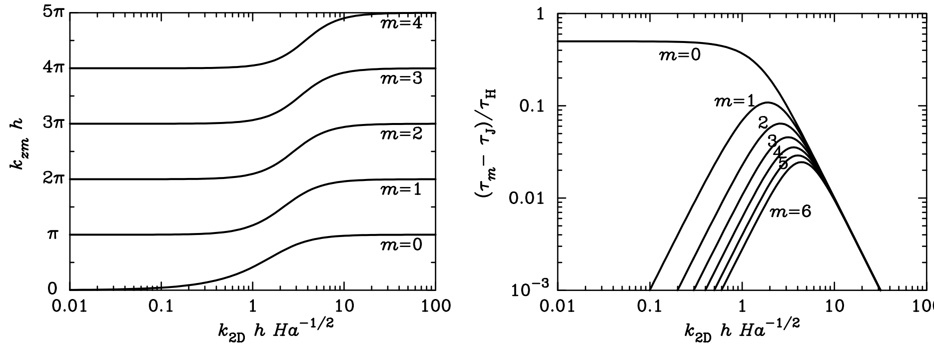


Fig. 2. Variations of the eigenvalues k_{zm} (left) and of the damping times τ_m (right) versus the wavenumber k_{2D}.

2. Damping properties. Fig. 2 show $k_{zm}h$ and $(\tau_m - \tau_J)/\tau_H$, where $\tau_H = (h/B_0)\sqrt{\rho/\sigma\nu}$ is the Hartmann damping time [1]. The time scale τ_0 is a monotonous decreasing function of k_{2D}, whereas the other time scales τ_m for $m \geq 1$ are non-monotonic functions of k_{2D} and have a maximum value at the wavenumber $k_{2D}^{\#} = [(m + \frac{1}{2})\,\pi\mathrm{Ha}]\,h^{-1}$. Across the wavenumber range around $k_{2D}^{\#}$, the eigenvalue k_{zm} increases from $m\pi$ to $(m+1)\pi$. We call $k_{2D}^{\#}$ the matching wavenumber. Furthermore, Table 1 provides approximations of the damping times τ_m, which are classified into several classes by the range of combination of k_{2D} and m. The results given in this table are discussed below for each class of vortices.

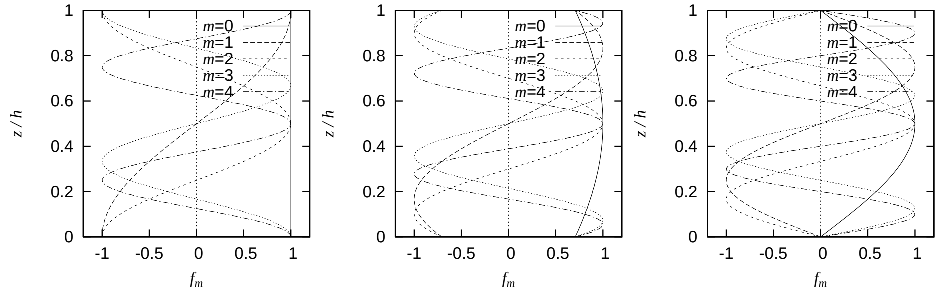
2.1. Large scale vortices: $k_{2D}h \lesssim 1$. Wavelength of vortices in this class is greater than the fluid layer depth h.Thus, all the higher mode vortices ($m \geq 1$) are oblate. The profile of the eigenfunction $f_0(z)$ is z-independent. The damping time τ_0 is a half of the Hartmann damping time τ_H, which is much longer than τ_J. These large-scale wall-to-wall vortices with long life-time are exactly 2D, as suggested by many experiments of quasi-2D MHD flow. The damping time of oblate vortices in this class ($m \geq 1$, $k_{2D}h \lesssim 1$) is the Joule time τ_J. It is much shorter than τ_H. Consequently, such vortices are rapidly damped out or do not exist from the beginning.

2.2. Medium scale vortices: $1 \lesssim k_{2D}h \ll [(m+\frac{1}{2})\pi\mathrm{Ha}]^{1/2}$. The wavelength of vortices in this class is smaller than h but greater than the matching scale. The profile of $f_0(z)$ is z-independent and τ_0 is a half of τ_H as well as the large

Table 1. Damping times τ_m and their classification.

	Large scale ($k_{2D}h \lesssim 1$)	Medium scale ($1 \lesssim k_{2D}h \ll k_{2D}^{\#}h$)	Matching scale ($k_{2D} \sim k_{2D}^{\#}$ a))	Tiny scale ($k_{2D} \gg k_{2D}^{\#}$)
Wall-to-wall ($m = 0$)	$\frac{1}{2}\tau_H$ b)	$\frac{1}{2}\tau_H$	$\frac{1}{\pi}\tau_H$	τ_ν c)
Prolate ($1 \leq m \ll k_{2D}h$)	—	$\left(\frac{k_{2D}h}{m\pi}\right)^2 \tau_J$	$\frac{1}{(2m+1)\pi}\tau_H$	τ_ν
Isotropic ($m\pi \sim k_{2D}h$)	—	$2\tau_J$	—	—
Oblate ($m \gg k_{2D}h$)	τ_J d)	τ_J	—	—

a) $k_{2D}^{\#} = \sqrt{(m+\frac{1}{2})\pi\mathrm{Ha}}/h$, b) Hartman damping time: $\tau_H = (h/B_0)\sqrt{\rho/\sigma\nu}$,
c) viscous damping time: $\tau_\nu = 1/(\nu {k_{2D}}^2)$, d) Joule time: $\tau_J = \rho/(\sigma {B_0}^2)$.



(a) (b) (c)

Fig. 3. Profiles of the eigenfunctions $f_m(z)$, (a) the medium scale vortices $1 \lesssim k_{2\mathrm{D}}h \ll [(m+\frac{1}{2})\pi\mathrm{Ha}]^{1/2}$, ($b$) the matching scale vortices $k_{2\mathrm{D}}h \sim [(m+\frac{1}{2})\pi\mathrm{Ha}]^{1/2}$, ($c$) the tiny scale vortices $[(m+\frac{1}{2})\pi\mathrm{Ha}]^{1/2} \ll k_{2\mathrm{D}}h \ll \mathrm{Ha}$.

scale vortices. The profiles of $f_m(z)$ for higher modes ($m \geq 1$) are sinusoidal functions with a maximum at the outer edge of the core flow as shown in Fig. 3a. If vortices are prolate ($1 \leq m \ll k_{2\mathrm{D}}h$), their damping time is much shorter than τ_{H} and much longer than τ_{J}. These vortices are the "quantized eddies" predicted by Sommeria and Moreau [1]. Quasi-2D MHD flow may be established after these vortices are damped out. If medium scale vortices are isotropic or oblate, their damping time is of the order of τ_{J}. Consequently, such vortices are rapidly damped out or do not exist from the beginning.medium scale $1 \lesssim k_{2\mathrm{D}}h \ll k_{2\mathrm{D}}^{\#}h$, because all higher-mode large-scale vortices ($m \geq 1$, $k_{2\mathrm{D}}h \lesssim 1$) are oblate and rapidly damped out in short duration τ_{J}.

2.3. Matching scale vortices: $k_{2\mathrm{D}}h \sim [(m+\frac{1}{2})\pi\mathrm{Ha}]^{1/2}$. The center of the wavenumber range of this class is the matching wavenumber $k_{2\mathrm{D}}^{\#} = [(m+\frac{1}{2})\pi\mathrm{Ha}]^{1/2}h^{-1}$. The wavelength of this class depends on the mode number m. It is of the same order as the classical parallel-layer when $0 \leq m \lesssim 10$. The damping times for $m = 0$, 1, 2 are of the same order as τ_0 of the large-scale wall-to-wall vortices. This fact suggests that actual quasi-2D MHD turbulent flows include not only $m = 0$ but also the higher mode ($m \geq 1$) eigenfunctions of the matching scale vortices. The profiles of $f_m(z)$ are shown in Fig. 3b. Even $f_0(z)$ has 30% variation along the magnetic field. Its "barrel-like" shape was predicted by Pothérat *et al.* [4].

2.4. Tiny scale vortices: $[(m+\frac{1}{2})\pi\mathrm{Ha}]^{1/2} \ll k_{2\mathrm{D}}h \ll \mathrm{Ha}$. The vortices in this class are characterized by the fact that τ_m is independent of the mode number m and $\tau_m = \tau_\nu = 1/(\nu {k_{2\mathrm{D}}}^2)$ is much smaller than τ_{H}. This suggests that the pre-existing tiny vortices are rapidly damped out.

One should also notice that $f_m(z)$ becomes zero at the outer edge of the core flow as shown in Fig. 3c. This suggests that these tiny vortices, if they exist, have no Hartmann layers and may be considered as ordinary hydrodynamic vortices rather than MHD vortices.

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