# HARTMANN EFFECT ON MHD TURBULENCE IN THE LIMIT $\mathrm{Rm} \ll 1$

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**Introduction.** Hartmann layers are very thin (the thickness may be around 30 $\mu$m in mercury with a magnetic field of 1 T), and except in those layers, where the shear is localized, the turbulence may be homogeneous. It is thus tempting to try to consider the core flow as a homogeneous system and model the effect of the Hartmann layers as extra terms in the evolution equations. Among the effects of the magnetic field on the turbulence, the quite first one is the development of an anisotropy, which becomes quite pronounced when the magnetic field is very large. In the case of homogeneous turbulence, this anisotropy is particularly clear in the Fourier space, since the Fourier transform of the Lorentz force depends on the direction of the wave vector, but is independent of its magnitude [1]. The energy carried by wave vectors parallel to the magnetic field $\mathbf{B}$ is rapidly damped out in a time scale of the order of $\tau_J = \rho/(\sigma B^2)$ ($\rho$ is the density, $\tau_J$ usually names the Joule time scale ). The net result of the competition between the Joule damping and inertia still leads to a time decay following a power law of the form $t^{-n}$ with $n \approx 1.7$ instead of 1.1 or 1.2 in ordinary isotropic turbulence [1]. In physical space, the anisotropic Lorentz force leads to the elongation of eddies in the direction parallel to $\mathbf{B}$ according to a law [4]

$$l_\parallel / l_\perp \sim \left(\sigma B^2 t/\rho\right)^{\frac{1}{2}}, \tag{1}$$

where $l_\parallel$ and $l_\perp$ are the length-scales of a given eddy parallel and perpendicular to the magnetic field. When the magnetic field is strong enough, eddies can elongate to sizes comparable to the size of the whole domain $h$. According to (1), this happens after a duration of order

$$\tau_{2D} = \tau_J \left(h/l_\perp\right)^2, \tag{2}$$

necessary to have $l_\parallel \sim h$. In that case, eddies are column-like and one may say that turbulence becomes Q2D. The effects we want to study here come from the influence of the Hartmann walls on the turbulence. This influence is at least twofold. First, the presence of the walls just forces the velocity component perpendicular to them to be zero, because it cannot vary significantly through the thin Hartmann boundary layer. Then, within the Q2D columns, the velocity vectors tend to remain within the planes perpendicular to the magnetic field. The second influence of the Hartmann walls has its origin in one of the striking properties of the Hartmann boundary layer, which is not a passive layer like the Blasius layer, but which is a primary layer, capable to react on the core flow. This property is the fact that a significant Joule damping remains present within the layer, where the velocity cancels, whereas the electric field $\mathbf{E}$ does not. As a consequence, the balance between $\mathbf{E}$ and $\mathbf{u} \times \mathbf{B}$, which is present in the core and minimizes the current density, is destroyed and the current density is locally very important (of the order of $\sigma B u$, whereas it is Ha less within the core, $\mathrm{Ha} = \sqrt{\sigma/(\rho\nu)} B h$ being the Hartmann number built with the width between the Hartmann walls $h$). On this basis, [4] have shown that the effective Joule damping time, which is then named the Hartmann time, is $\tau_H = \mathrm{Ha}\,\tau_J = h/B\sqrt{\rho/(\sigma\nu)}$. It may be much larger than $\tau_J$, since it varies as $B^{-1}$ (not $B^{-2}$ as $\tau_J$).

**1. Model formulation.** According to what was described in the above section, the turbulence can become Q2D as soon as $\tau_{2D}$ is significantly shorter than the eddy turnover time $\tau_{tu} = l_\perp/u_\perp$. Then the velocity component parallel to the magnetic field, which may be non-zero in the initial state, is submitted to a linear damping in a time scale of the order of $\tau_{2D}$. This may be easily modeled with the addition of the term $-u_\parallel/\tau_{2D}$ in the right-hand side of the equation for $u_\parallel$. The velocity components in the plane perpendicular to **B** are also affected by some damping, but much less rapidly, as explained in Sec. 2. Indeed, they are only submitted to the Hartmann damping, which may be expressed by the addition of the term $-u_\perp/\tau_H$ in the right- hand side of the equation for $u_\perp$. The damping force $F_i$ associated with the Hartmann layers is, therefore, tentatively modeled by

$$F_x = -u_x/\tau_H,\; F_y = -u_y/\tau_H,\; F_z = -u_z/\tau_{2D}, \tag{3}$$

where the $z$-direction has been chosen as the wall-normal direction. The form (3) cannot be used as such since it would not respect the incompressibility of the flow. However, it can easily be projected on its solenoidal part $F_i^S$. Using the Fourier representation, $F_i^S$ can be written as

$$F_i^S = (\delta_{ij} - \frac{k_i k_j}{k^2})F_j. \tag{4}$$

Again in Fourier representation, the actual components of $F_i^S$ are then,

$$F_x^S = -u_x/\tau_H + (1/\tau_{2D} - 1/\tau_H)\frac{k_x k_z}{k^2}u_z \tag{5}$$

$$F_y^S = -u_y/\tau_H + (1/\tau_{2D} - 1/\tau_H)\frac{k_y k_z}{k^2}u_z \tag{6}$$

$$F_z^S = -u_z/\tau_{2D} + (1/\tau_{2D} - 1/\tau_H)\frac{k_z k_z}{k^2}u_z \tag{7}$$

Aside from a simple linear damping, incompressibility thus requires also a wave vector dependent contribution in order to take into account the effect of Hartmann layers. In (7), we see that this $k$-dependent contribution has exactly the same functional form as the traditional Joule damping term. It comes, however, with the opposite sign. When $\tau_{2D}$ and $\tau_J$ are of the same order, the traditional Joule damping can, therefore, be compensated by this extra term (since in general $\tau_H \gg \tau_J$) and it is thus expected that anisotropy in the parallel component will remain weak in that case. Its decay will be dominated by the simple damping term $-u_z/\tau_{2D}$ at all times. For the perpendicular components, two phases of decay have to be distinguished. In the first one, $0 < t < \tau_{2D}$, the $k$-dependent terms will exert their effects. In the second one, $\tau_{2D} < t$, their influence will become negligible as $u_z$ is damped very rapidly. In this second phase, the decay will proceed as a simple damping with a characteristic time $\tau_H$.

**2. Numerical results.** The set of equations we are solving are:

$$\partial_t u_i = -\partial_i(p/\rho) - u_j\partial_j u_i - \frac{A^2}{\eta}\Delta^{-1}\partial_z\partial_z u_i + F_i^S + \nu\Delta u_i, \tag{8}$$

where $F_i^S$ is given by (5)–(7) (here$A = B/\sqrt{\mu\rho}$ denotes the Alfven velocity).These equations are solved using a pseudo-spectral code in a cubic geometry. There solutions of our run are $256^3$ Fourier modes. The initial condition for the velocity consists of a developed turbulence field that is adequately resolved in the computational domain adopted (it is obtained from a purely hydrodynamic case). Some of its characteristics are listed in the Table 1.

*Table 1.* Turbulence characteristics of the initial velocity field. All quantities are in MKS units.

| | |
|---|---|
| Resolution | $256^3$ |
| Box size ($l_x \times l_y \times l_z$) | $(2\pi)^3$ |
| Rms velocity | 2.35 |
| Viscosity | 0.006 |
| Integral length-scale ($L = 3\pi/4 \times (\int \kappa^{-1} E(\kappa) \mathrm{d}\kappa / \int E(\kappa)\mathrm{d}\kappa)$) | 0.944 |
| $\mathrm{Re} = uL/\nu$ | 370 |
| Dissipation ($\epsilon$) | 14.56 |
| Dissipation scale ($\gamma = (\nu^3/\epsilon)^{1/4}$)) | 0.0110 |
| $k_{\max}\gamma$ | 1.41 |
| Microscale Reynolds number ($\mathrm{R}_\lambda = \sqrt{15/(\nu\epsilon)}u^2$) | 72.36 |
| Eddy turnover time ($\tau = L/u$) | 0.402 |

In order to induce a sufficient amount of anisotropy in the flow, we have chosen a moderate value of the interaction number: $N = \tau_{tu}/\tau_J = 10$. Giving the values of $u$ and $L$ (see Table 1) implies that the Joule time is equal to $\tau_J = L/(uN) = 0.0402$. In this run we also assume that $\tau_{2D} = 0.0402 = \tau_J$ (looking at (2) this corresponds to a case, where the channel width is equal to the initial $l_\perp$). From these values, one also easily computes that $\mathrm{Ha} = 60.8$ and $\tau_H = 2.45$.

*2.1. Decay of kinetic energy.* At the beginning of the simulation all three components of the velocity field have approximately equal energy (the initial flow is isotropic). As displayed in Fig. 1, the energy of the parallel component is dissipated much faster than the one of the perpendicular components (since $\tau_{2D} \ll \tau_H$). After a time $t = 2\tau_{2D}$, the parallel component has been virtually completely dissipated.

*2.2. Anisotropy.* To measure anisotropy, we use the following diagnostics:

$$G_{ij,kl} = \left\langle (\partial_i u_j)^2 \right\rangle / \left\langle (\partial_k u_l)^2 \right\rangle. \tag{9}$$

They measure the relative strength of velocity gradients in different directions and for different components of the velocity field. For instance, it is easy to show that in isotropic turbulence one should have ([3]):

$$G_{11,21} = G_{11,31} = 0.5,\ G_{12,22} = G_{13,33} = 2,\ G_{12,32} = G_{13,23} = 1. \tag{10}$$

In our simulations, the mean magnetic field is directed along the $z$-direction (parallel direction). If the flow becomes 2D perpendicular to this direction, then one must have:

$$G_{3j,\alpha l} \to 0, \quad \text{for} \quad (j,l) \in (1,2,3) \quad \text{and} \quad \alpha \in (1,2). \tag{11}$$

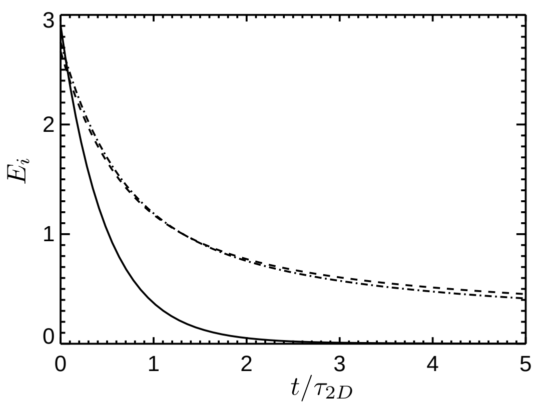


*Fig. 1.* Evolution with time of the kinetic energy. Solid line: $\langle \|\frac{1}{2}u_z\|^2 \rangle$; dashed line: $\langle \|\frac{1}{2}u_x\|^2 \rangle$; dashed-dot line: $\langle \|\frac{1}{2}u_y\|^2 \rangle$.

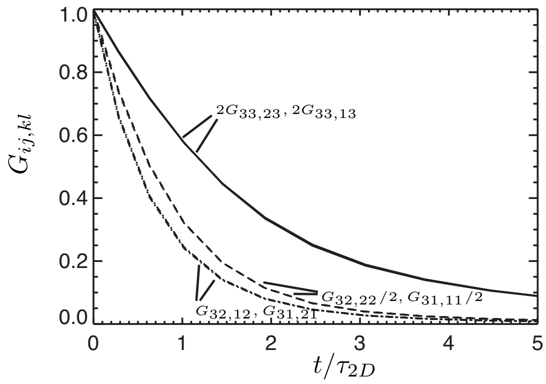


*Fig. 2.* Anisotropy coefficients $G_{ij,kl}$. See figure for legend.

In Fig. 2 several $G_{ij,kl}$ are plotted. From the figure it is clear that the symmetry $x \leftrightarrow y$ is well respected. Also, the component $u_z$ remains significantly more isotropic than the perpendicular components with respect to this diagnostics, although with time it also evolves to a state with weaker variations along the parallel direction compared to the perpendicular directions. This behavior is well in line with the discussion in Sec. 1. On dimensional grounds, all $G_{ij,kl}$ displayed in Fig. 2 should be of the order $l_\perp^2/l_\parallel^2$. Here we see that this ratio depends on the component of the velocity field considered. The ratio is larger for the parallel component than for the perpendicular components. As a consequence, we see that Eq. (1) becomes component-dependent in the presence of the Hartmann damping modeling (5)–(7).

**3. Conclusions.** The idea of a pre-existing isotropic turbulence suddenly submitted to a uniform magnetic field, initially introduced by [2], is a purely idealized concept, since, in any experiment, during the growth of the magnetic field, eddy currents and the associated Lorentz forces are generated, which are completely neglected here. In spite of this assumption, which would deserve quite a complex analysis and may be not justified at all in many experimental situations, following Moffatt and others, we focus on the mechanisms, by which the homogeneous turbulence tends to become 2D and decays. Contrary to previous numerical studies on 3D homogeneous turbulence subject to a constant mean magnetic field, we try to incorporate the influence of distant Hartmann layers on the flow. The spirit behind the model introduced in Sec. 1 is to reproduce the damping generated by the Hartmann layers though damping terms of appropriate time scales. However, to respect the incompressibility condition, a wave vector dependent contribution to the model have to be considered. The effect of the model is, therefore, twofold. First, the velocity component parallel to the magnetic field undergoes a much more rapid damping than the perpendicular components. Less obviously, this parallel component remains largely isotropic throughout the decay due to the wave vector dependent contribution in the model.

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