MHD INSTABILITIES IN HYDROMAGNETIC TAYLOR-COUETTE FLOWS

G. Rüdiger

Astrophysikalisches Institut Potsdam,
An der Sternwarte 16, D-14482 Potsdam, Germany

The linear instability of MHD Taylor–Couette flows is considered. The magnetic field destabilizes the rotating flow by the magnetorotational instability (MRI). The latter exists for hydrodynamically unstable flows only for high enough magnetic Prandtl number Pm. It is more pronounced for hydrodynamically stable flows. One can always find a field amplitude, where the characteristic Reynolds number is minimum. In all cases the minimum magnetic Reynolds number is of the order 10 and the Lundquist number slightly exceeds unity. The critical Reynolds numbers thus exceed the values of 10^6 for sodium and 10^7 for gallium. The numbers are drastically reduced if an extra toroidal magnetic field is applied with an amplitude larger than the amplitude of the axial field.

Introduction. The rotation law in the infinite Taylor–Couette flow is $\Omega(R) = a + b/R^2$, where a and b are related to the angular velocities $\Omega_{\rm in}$ and $\Omega_{\rm out}$ of the inner and the outer cylinders. With $R_{\rm in}$ and $R_{\rm out}$ as the radii of the cylinders, the parameters $\hat{\mu} = \Omega_{\rm out}/\Omega_{\rm in}$ and $\hat{\eta} = R_{\rm in}/R_{\rm out}$ of the flow are defined. The ideal flow is hydrodynamically stable when $\hat{\mu} > \hat{\eta}^2$. If the fluid is electrically conducting and an axial magnetic field is applied, then after the old results the critical Reynolds number (for the inner rotating cylinder) grows with the growing magnetic field (Chandrasekhar 1961). Velikhov (1959), however, discovered a magnetic shear-flow instability, which is now called 'magnetorotational instability' (MRI). He found that for the ideal hydromagnetic Taylor–Couette flow the Rayleigh criterion for stability changes to $\hat{\mu} > 1$ and that

$$k \le 2\hat{\eta}\frac{\Omega_{\rm in}}{V_{\rm A}}, \qquad (1)$$

is the critical wave number ($V_{\rm A}$ Alfvén velocity). The MRI reduces the critical Reynolds number for weak magnetic field strengths for the hydrodynamically unstable flow and it destabilizes the otherwise hydrodynamically stable flow for $\hat{\eta}^2 < \hat{\mu} < 1$. As we shall demonstrate, the magnetic Reynolds number $\mathrm{Rm} = \mathrm{Re\ Pm}$ mainly controls the instability. Because of the high value of η for liquid metals (exceeding 1000 cm^2/s), it is not easy to reach magnetic Reynolds numbers of the required order of $1 \ldots 10$. This is the basic reason why the MRI has not yet been observed in the laboratory.

Taylor–Couette flow under the presence of an axial magnetic field. The Reynolds number is usually defined as $\mathrm{Re} = R_{\rm in}(R_{\rm out} - R_{\rm in})\Omega_{\rm in}/\nu$. The amplitude of the external magnetic field B_0 is expressed by the Hartmann number $\mathrm{Ha} = B_0\sqrt{R_{\rm in}(R_{\rm out} - R_{\rm in})/(\mu_0\rho\nu\eta)}$. Fig. 1 shows the neutral stability for axisymmetric modes for containers with a resting outer cylinder and for various magnetic Prandtl numbers. $\mathrm{Re} = 68$ is the classical hydrodynamic solution for resting outer cylinder and $\hat{\eta} = 0.5$. Note the strong difference of the bifurcation lines for $\mathrm{Pm} \gtrsim 1$ (high conductivity) and $\mathrm{Pm} < 1$ (low conductivity). For fluids with low a electrical conductivity the magnetic field only suppresses the instability so that all the critical Reynolds numbers strongly exceed the value 68.

The opposite is true for $\mathrm{Pm} \gtrsim 1$. In Fig. 1 the resulting critical Reynolds numbers Re are smaller than 68. The magnetic fields with small Hartmann numbers

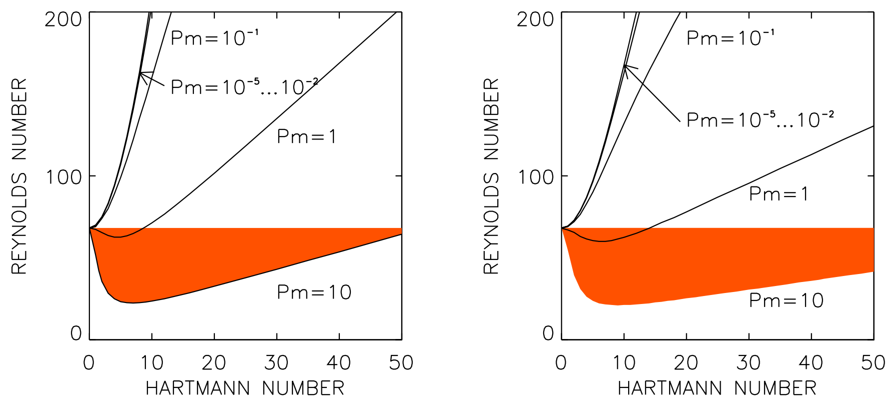

Fig. 1. Bifurcation diagram for axisymmetric modes with a resting outer cylinder of conducting material (left) and vacuum (right). Shaded areas denote subcritical excitation by the axial magnetic field. $\hat{\eta} = 0.5$. From Rüdiger, Schultz & Shalybkov (2003).

support the instability rather than suppress it. This effect becomes more effective for an increasing Pm but it vanishes for stronger magnetic fields.

Now the outer cylinder may rotate so fast that the rotation law no longer fulfills the Rayleigh criterion. The nonmagnetic eigenvalue along the vertical axis moves to infinity but a minimum remains. Fig. 2 presents the results for Pm = 1 and Pm $= 10^{-5}$ (Rüdiger & Shalybkov 2002). For $\hat{\eta} = 0.5$ and $\hat{\mu} = 0.33$ the critical Reynolds numbers together with the critical Hartmann numbers are plotted in Fig. 3. The general scaling $\mathrm{Re} \propto \mathrm{Pm}^{-1}$ results leading to

$$\mathrm{Rm} \simeq \mathrm{const.} \qquad \mathrm{S} \simeq \mathrm{const.} \tag{2}$$

for the magnetic Reynolds number Rm and the Lundquist number $\mathrm{S} = \mathrm{Ha}\sqrt{\mathrm{Pm}}$. For hydrodynamically unstable flows the cell has the same vertical extent as it has in radius. The magnetic field deforms the Taylor vortices elongating the cell in the vertical direction. The wavenumber is thus expected to become smaller and smaller for the increasing magnetic field. This is indeed true (Fig. 4).

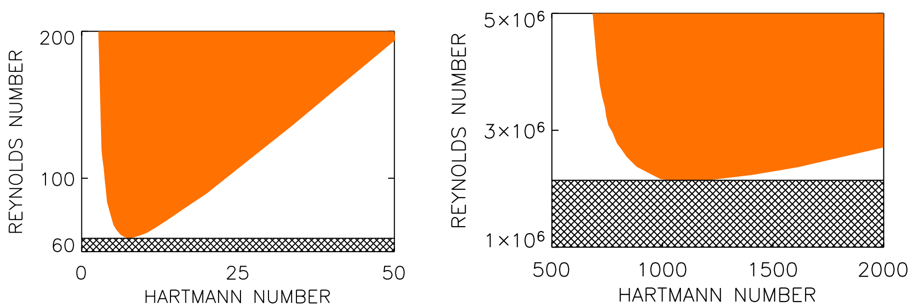

Fig. 2. Marginal stability lines for axisymmetric modes in containers with a rotating outer cylinder of conducting material for Pm = 1 (left) and Pm $= 10^{-5}$ (right). $\hat{\eta} = 0.5, \hat{\mu} = 0.33$. The instability domain is grey-colored and fluids in the cross-hatched area are always stable.

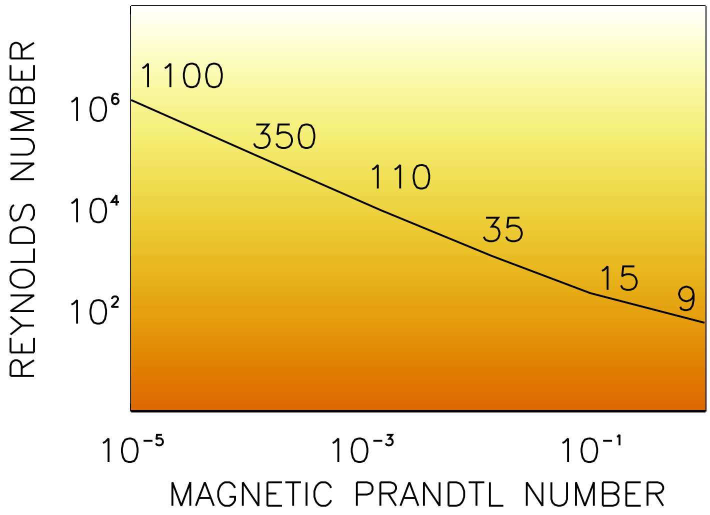

Fig. 3. The critical Reynolds numbers vs. the magnetic Prandtl numbers marked with those Hartmann numbers where, the Reynolds number is minimum. $\hat{\eta} = 0.5$, $\hat{\mu} = 0.33$.

Taylor–Couette flow under the presence of a spiralic magnetic field. The given solutions of the marginal stability of axisymmetric modes are stationary. One could ask for the character of the solutions if an extra toroidal magnetic field is applied (Hollerbach & Rüdiger 2005). We know that the current-free toroidal fields alone do not change the stability of Taylor–Couette flow (Velikhov 1959). If axial fields *and* current-free toroidal fields of the same order exist in the container, however, completely new solutions appear and they are oscillating (Fig. 5). The

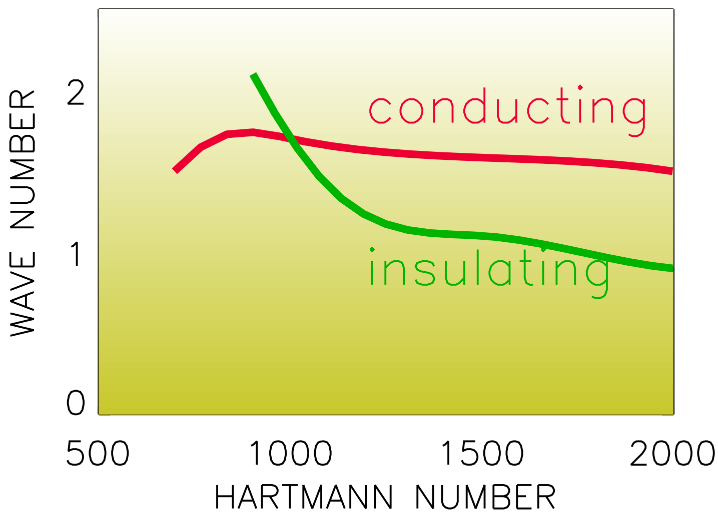

Fig. 4. Wave numbers, for which the Reynolds number is minimum for conducting walls and insulating walls. (Pm $= 10^{-5}$, Rüdiger & Shalybkov 2002).

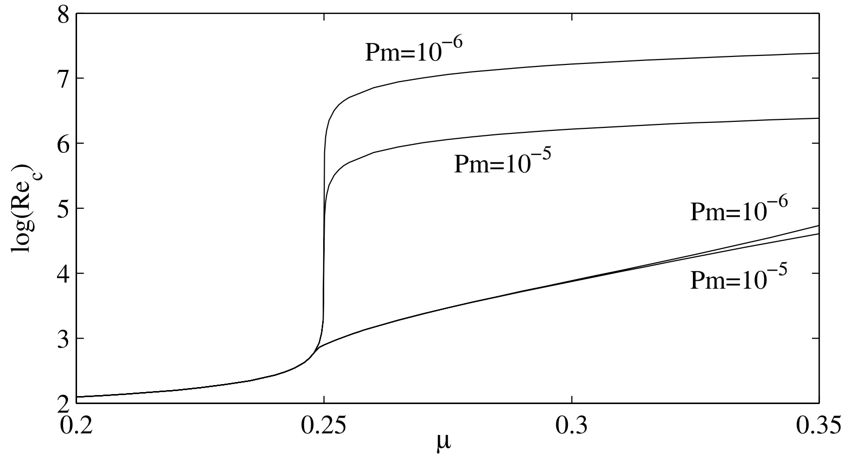

Fig. 5. Critical Reynolds numbers in containers ($\hat{\eta} = 0.5$) with a purely axial magnetic field (upper curves) and with an extra toroidal field (lower curves) for Pm $= 10^{-5}$ (sodium) and Pm $= 10^{-6}$ (gallium). The cylinders are insulators. The Prandtl number dependence of the lower curves is extremely weak and the Reynolds numbers are strongly reduced (Hollerbach & Rüdiger 2005).

real parts of the eigenfrequencies are positive so that stationary modes do not longer exist. The Reynolds number of the flow and the Hartmann number of the axial field, which are necessary to induce the MRI instability, are reduced by orders of magnitude by the toroidal field compared to the case of purely axial fields. In such experiments the toroidal magnetic field can be produced by axial currents within the inner cylinder. The amplitude of the currents is of the order of 10^3 A.

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