

INTERMITTENCY IN THE HOMOPOLAR DYNAMO

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Introduction. The Bullard (or homopolar) dynamo [1] is the first example of a magnetic instability triggered by a conductor in motion. The experimental device is depicted in Fig. 1: a conducting disk rotates around its axis at an angular velocity ω and a small magnetic field is applied in the vertical direction which induces a current in the disk from the axis towards the edge of the conducting cylinder. Then this current flows in the spire which is wound around the axis of the cylinder. If the spire is orientated in a special sense, the induced magnetic field is orientated in the same direction as the initial one. Thus an infinitesimal magnetic field can be amplified by this mechanism, leading to a dynamo.

We here study a slight modification of the original Bullard dynamo, where azimuthal currents in the disk and mechanical friction are permitted. The azimuthal current has been introduced by [2] who showed that the hypothesis of a purely radial current would violate the conservation of magnetic flux in the case of a perfect conducting cylinder. Three equations are enough to describe this system [3]:

$$\begin{cases} \dot{x} = q(y - x) \\ \dot{y} = xZ + mx - (m + 1)y \\ \dot{Z} = g[1 - (m + 1)xy + mx^2] - z, \end{cases} \quad (1)$$

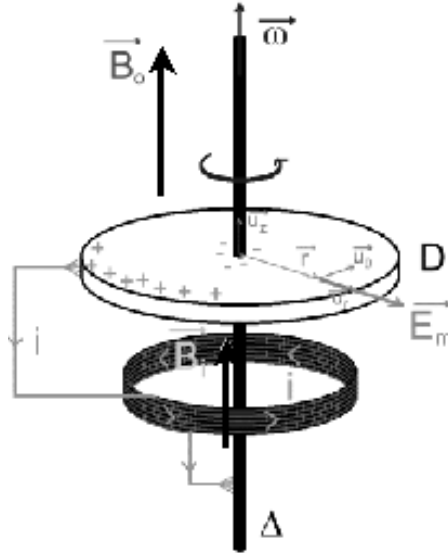


Fig. 1. The homopolar (or Bullard) dynamo.

where Z is the adimensionalized angular velocity of the disk and x and y are the two components of the magnetic flux, across the disk and across the spire. The two first equations represent the effect of the rotating cylinder on the magnetic field and the third equation is the evolution of the angular velocity, subject to a Joule heating (the term proportional to g) and a mechanical torque (the last term). A linear instability analysis around the solution $Z = Z_0$ (given angular velocity) and $x = y = 0$ proves that the system becomes unstable as soon as $Z_0 > 1$.

1. Stochastic analysis of the system. To study this system, we make a kinematic approximation. We prescribe the velocity field and study the generation of the magnetic field induced by this given velocity. Specifically, we decompose the velocity field as a sum of a mean part and a fluctuating part: $Z = Z_0 + \Gamma(t)$. For simplicity, we assume that the fluctuating part of the velocity field is white in time : $\langle \Gamma(t)\Gamma(t') \rangle = 2D\delta(t - t')$. Applying this procedure to the two first equation of (1), we are left with a linear stochastic system. It is well known that it is difficult to define unambiguously a threshold for this system and it is necessary to account for the non-linearity induced by the third equation. In consequence, we introduce a non-linear term $-(m + 1)gxy$ to mimic the retroaction of the variables x and y on the intensity of the adimensionalized angular velocity.

To study (1), we make the following change of variables from the Cartesian coordinates (x, \dot{x}) to the polar ones (r, θ) :

$$x = r \cos(\theta) \quad \text{and} \quad \dot{x} = r \sin(\theta) \tag{2}$$

and we derive an equation for the probability density function of having a couple (r, θ) . To solve this problem, we further assume that this distribution can be separated in one part depending only on the radial variable $P(r)$ and one depending only on the angular one $I(\theta)$.

It is then possible to find the stationary probability density of the radial variable:

$$P_s(r) = \frac{1}{Z} r^{b/a-1} \exp \left[-\frac{cr^2}{2a} \right] \quad \text{with} \quad Z = \frac{1}{2} \left[\frac{2a}{c} \right]^{b/(2a)} \Gamma \left(\frac{b}{2a} \right), \tag{3}$$

where a , b and c are the functions of the parameters m , q , Z_0 , g and D and also from the angular distribution $I(\theta)$.

When b is negative, one can check that the distribution (3) is not integrable in zero. In that case, the only admissible solution is a Dirac function centered around zero: $P_s(r) = \delta(r)$, which corresponds to a solution $x = 0$ at long time. Indeed, one can check that it is always a solution of equation (1). Then, we can identify two bifurcations whether the control parameter is taken to be the mean value of r or its most probable value. This scenario has already been evidenced in the case of a stochastic modeling of the dynamo effect [4]. To compute the threshold value corresponding to these two bifurcations, one needs to characterize completely the probability density (3) and thus to calculate the coefficient defined above. To achieve that, we calculated the density probability $I(\theta)$ in a similar way as [5].

2. Bifurcation scenario. The bifurcation of the system (1) is of a rather complex nature. Indeed, depending on the control parameter, two thresholds can be identified. This is to be contrasted with the case without noise, where $Z_0 < 1$ corresponds to an absorbing state, where the system relaxes towards $x = y = 0$ and $Z_0 > 1$, where the solution at long time converges to a non zero value. We now characterize more deeply the nature of the bifurcation when the noise intensity is not zero.

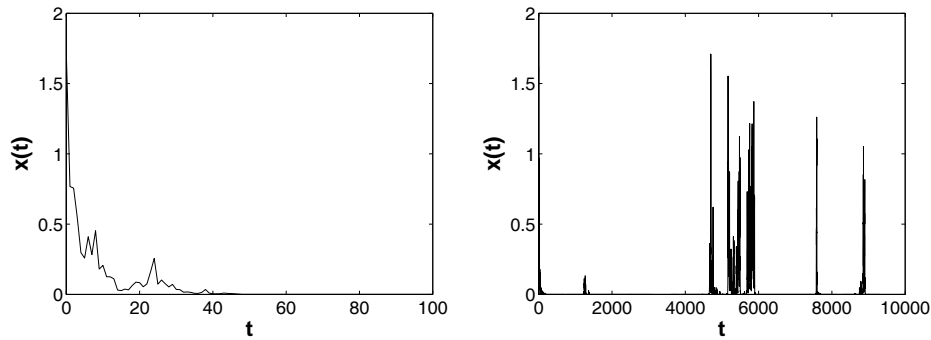


Fig. 2. Result of a numerical simulation of system (1) with $m = 1$, $q = 1$, $g = 1$, and $D = 27$. The left panel is for $Z_0 = -0.8$, corresponding to an absorbing state, and the right panel for $Z_0 = 2.8$, corresponding to an intermittent state.

2.1. $b > 0$: bifurcation towards an intermittent state. When b becomes positive, the distribution (3) is integrable near the origin and can thus represent a meaningful probability distribution function. However, the distribution diverges in zero, an indication that the most probable value is still zero. This is characteristic of an intermittent state, where the signal exhibits bursts of activity separated by quiescent epochs, where the norm is close to zero. To illustrate such a behavior, we performed numerical simulations of equation (1) and some typical snapshot are shown in Fig. 2.

On the left-hand side, we see that the variable x after a short transient time (compare the time interval of the two snapshots) relaxes towards the absorbing state. Increasing the parameter Z_0 (keeping the other parameter fixed at the value of Fig. 2), one notices that a first bifurcation occurs (around $Z_0 = 2.5$) which leads to a state, where the variable x is most of the time close to zero but exhibits bursts (cf. the right hand side of the figure). We call this type of behavior intermittency.

To characterize the intermittent state, we use the time series of x to compute the different parameters that appeared in the previous section a , b , c . Once this parameters are known, we can compute the probability distribution of the variable r and θ . The theoretical distributions show a good agreement with the numerical ones as it is shown in Fig. 3.

2.2. $b > a$: bifurcation towards a turbulent state. When b becomes larger than a , one can easily check that the distribution has now a well defined maximum

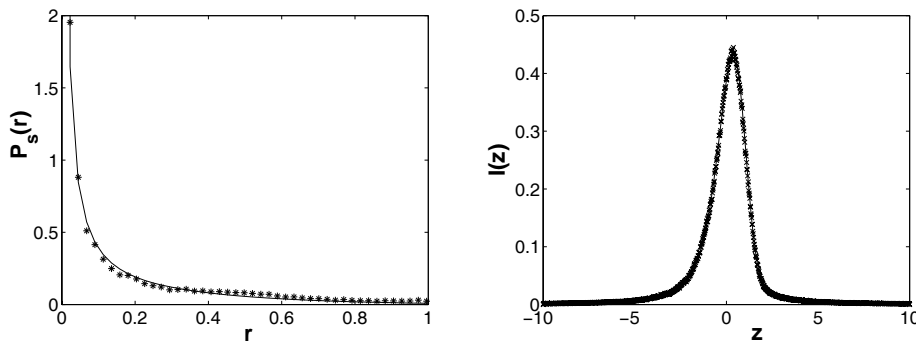


Fig. 3. Comparison of the simulation with $m = 1$, $q = 1$, $g = 1$, $D = 27$ and $Z_0 = 2.8$ corresponding to the intermittent state and the analytical prediction (in straight line) for the probability density of the variable r (left-hand side) and $z = \tan \theta$ (right-hand side). The parameters have been found numerically $a = 0.14$, $b = 0.019$, $c = 0.47$.

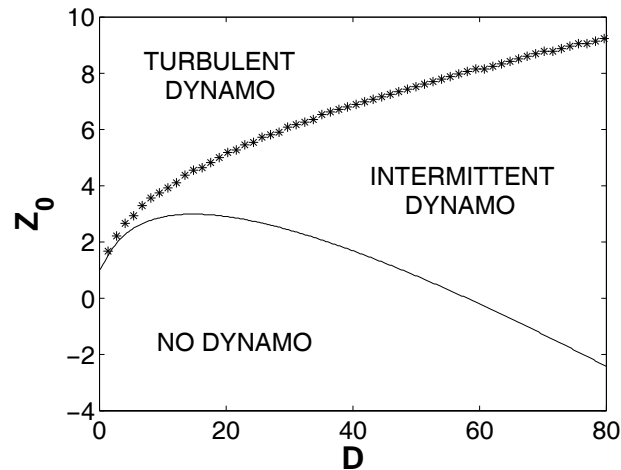


Fig. 4. Bifurcation diagram of the system (1).

for $r = \sqrt{(b-a)/c}$. In this new regime, the variable $x(t)$ fluctuates around a well defined most probable value.

3. Stability Diagram. Using, the expression for the probability distribution of θ , one may then express the condition of instability $b > 0$ and find the bifurcation line $Z_0^*(D)$. The numerical integration of this condition is drawn in Fig. 4 as a straight line. This curve delineates the parameter space between the absorbing state (or “no dynamo” state), corresponding to x going to zero for a long time and the (intermittent and turbulent) dynamo states.

The result of the numerical integration of the condition $b > a$ is shown in Fig. 4 (with *). One sees that the evolution of this threshold is monotonous: when the noise is increased, the transition from the intermittent to the turbulent state is delayed. On the contrary, the transition from the absorbing state to the intermittent one is first increased for weak intensities of the noise (stabilization by noise) and then is lowered below its deterministic value for a more powerful noise (a reentrant transition that has been pointed out by [5]).

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